Advanced field theory: Supersymmetric black holes and the black hole attractor mechanism

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Contents

Information on exercises:
Exercises of Sections 22.5 – 22.8 of the *Supergravity* book [1], including some of the optional 'level 3' exercises, are displayed in grey boxes. An overview of the exercises and the pagenumber at which they start is given below. Exercise numbers are clickable links for PDF readers that allow the reader to jump to the exercises.

1 Introduction

Black hole attractors are solutions of supergravity theories containing gauge and scalar fields, and carry an electric as well as magnetic charge. Such solutions appear in various extended supergravity theories, but here we will only discuss some examples in $\mathcal{N} = 2, D = 4$ supergravity coupled to a single vector multiplet only. Furthermore, we limit our analysis to extremal black hole solutions and will always assume our space-time to be static and spherically symmetric. Interesting physical features we will discuss are the attractor mechanism, residual supersymmetry in the black hole solutions and a set of first-order differential equations, exhibiting a gradient flow, which describe the scalar dynamics.

The values of the scalar fields at spatial infinity determine the mass of the black hole. However, any value of the scalar field at spatial infinity will have a fixed value at the horizon of the black hole. This value only depends on the charges of the black hole, and the 'memory' of the scalar fields' value at infinity is lost. The name 'attractor' refers to the set of first-order differential equations that govern the 'flow' of the scalars, reminiscing of a gradient flow towards fixed points from dynamical systems theory. We will see that the area of the horizon and hence the black hole's entropy depends only on the charges of the black hole and is independent of the boundary conditions on scalars.

These black hole attractors also arise as solutions of vanishing linearized fermion supersymmetry transformations of $\mathcal{N} = 2$ supergravity, implying they are supersymmetric. The concept of central charges in extended supersymmetry will play a crucial role in our analysis of both the attractor mechanism, as well as proving that our solutions are supersymmetric, and this framework allows one to easily generalise our work to include an arbitary number of vector multiplets.

The report is organized as follows. In Section 2, we recall a few concepts on black holes and electromagnetic dualities. In Section 3, we consider a first example of a black hole attractor: the dilaton black hole. Section 4 sets the scene for more general black hole attractor solutions. Supersymmetric solutions and central charges are introduced in Section 5, where we also show that the dilaton black hole is supersymmetric. After this, we return to the more general solutions and show that they are attractors in Section 6. The attractor mechanism allows us to easily show they are supersymmetric as well. Section 7 gives a few remarks on generalisations to an arbitrary amount of vector multiplets and to other theories or solutions. Besides this, we briefly discuss how supersymmetry can act as a cosmic censor. The report tries to be as self-contained as possible, but some results are taken without proof from the *Supergravity* book [1], from now on referred to as 'the book'.

2 Preliminaries

In this section, we repeat a few basic results which are covered in the *Advanced Field Theory* and/or other courses which are needed to understand the main body of the text.

2.1 Recap on black hole solutions

When looking for general static and spherically symmetric solutions, one ends up with black holes as an interesting class of possible solutions. Here we briefly repeat a few basic concepts on charged black holes, which are useful to understand the report.

The *Reissner-Nördstrom* metric in 4-dimensional space-time describes a non-rotating, *dyonic* (electrically and magnetically charged) black hole and is given by

$$\mathrm{d}s^2 = -F(r)\,\mathrm{d}t^2 + \frac{1}{F(r)}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega_2^2\,, \tag{2.1}$$

where $\mathrm{d}\Omega_2^2 \equiv \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2$ is the line element of the 2-sphere. The function $F(r)$ is given by

$$F(r) = 1 - \frac{2MG}{r} + \frac{(q^2+p^2)G}{4\pi r^2}\,, \tag{2.2}$$

where q, p are the electric and magnetic charge, respectively, while M is the mass of the black hole. At $r = 0$, this solution has a true *singularity*, which cannot be remedied by coordinate transformations. A black hole's *event horizon* (or simply horizon) is roughly speaking a surface past which radially infalling particles can never escape to infinity [2]. In our case, these surfaces are 2-spheres with radii equal to the zeroes of $F(r)$, which are

$$r_\pm = MG \pm \sqrt{M^2G^2 - \frac{(q^2+p^2)G}{4\pi}}\,. \tag{2.3}$$

A *naked singularity* is a singularity which is not hidden behind an event horizon. The cosmic censorship conjecture dictates that solutions with naked singularities cannot form in gravitational collapse. For the Reissner-Nördstrom black hole, this implies a lower bound on the mass such that $r_\pm$ are both real:

$$M^2 \geq \frac{q^2+p^2}{4\pi G}\,. \tag{2.4}$$

When the bound is satisfied, the black hole is said to be *extremal.* In this case, $r_+ = r_- = r_S$, such that there is a single horizon, located at the Schwarzschild radius $r_S = MG$. The metric then becomes

$$\mathrm{d}s^2 = -(1 - MG/r)^2\,\mathrm{d}t^2 + (1 - MG/r)^{-2}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega_2^2\,. \tag{2.5}$$

In our treatment, we prefer to have the horizon at radial coordinate $r = 0$, such that we introduce a new radial coordinate $v = r - MG$ and the metric is

$$\mathrm{d}s^2 = -(1 + MG/v)^{-2}\,\mathrm{d}t^2 + (1 + MG/v)^2\left(\mathrm{d}v^2 + v^2\,\mathrm{d}\Omega_2^2\right)\,. \tag{2.6}$$

Near the horizon $v = 0$, we can expand the metric and find

$$\mathrm{d}s^2 \approx -\frac{v^2}{(MG)^2}\,\mathrm{d}t^2 + (MG)^2\frac{\mathrm{d}v^2}{v^2} + (MG)^2\,\mathrm{d}\Omega_2^2\,. \tag{2.7}$$

By introducing $z = (MG)^2/v$, this can also be written as

$$\mathrm{d}s^2 \approx \frac{(MG)^2}{z^2}\left(-\,\mathrm{d}t^2 + \mathrm{d}z^2\right) + (MG)^2\,\mathrm{d}\Omega_2^2\,, \tag{2.8}$$

which is known as the *Robinson-Bertotti metric.* It describes the product space $\mathrm{AdS}_2 \times S^2$, where the AdS space L and radius of the sphere r are both equal to r_S.

2.2 Recap on electromagnetic duality

Electromagnetic duality is an interesting symmetry of field theories which contain abelian gauge fields and that possibly interact with other fields. At the heart of this duality lies the concept of the *dual tensor*, which for antisymmetric tensors $F_{\mu\nu}$ of rank 2 in 4D Minkowski space-time is defined as

$$\tilde{F}_{\mu\nu} \equiv -\tfrac{1}{2} i \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \,, \tag{2.9}$$

where $\varepsilon_{\mu\nu\rho\sigma}$, the Levi-Civita tensor, is defined with convention $\varepsilon_{tr\theta\varphi} = 1$. The square of the tilde operation is the identity. The relation between the tilde operation and the Hodge dual is given by

$$\tilde{F}_{\mu\nu} = -i \left({}^*F\right)_{\mu\nu} \tag{2.10}$$

For field strength 2-forms in $D = 4$, the components of ${}^*F$ are

$${}^*F_{\mu\nu} = \tfrac{1}{2}\sqrt{-g}\,\varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} \,, \quad {}^*F^{\mu\nu} = \frac{1}{2\sqrt{-g}} \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \,. \tag{2.11}$$

We can furthermore define the 2-forms

$$F^{\pm}_{\mu\nu} \equiv \tfrac{1}{2}(F_{\mu\nu} \pm \tilde{F}_{\mu\nu}) \,, \tag{2.12}$$

which often appear in formulae of $\mathcal{N} = 2$ supergravity. For a free abelian gauge field, the Maxwell and Bianchi equations are neatly summarized if we exploit the dual tensors: they become

$$\partial_\mu F^{\mu\nu} = 0 \,, \quad \partial_\mu \tilde{F}^{\mu\nu} = 0 \,. \tag{2.13}$$

For this reason, we can think of $F_{\mu\nu}$ as the basic field variable and ignore the underlying vector potential. This is the approach we will adopt in this work. The change of variables

$$F^{\mu\nu} \to F'^{\mu\nu} = i \tilde{F}^{\mu\nu} \tag{2.14}$$

is a symmetry of the free abelian gauge field. The symmetry exchanges electric and magnetic fields.

The simplest extension to interacting field theories is the case where the abelian gauge field is coupled to a complex scalar field. This is common in supergravity theories, such as actions where the kinetic terms of gauge fields depend on scalar fields. The duality transformation, with the additional scalar field present, must now be extended such that all equations of motion remain invariant. It is useful to introduce the real tensor

$$G^{\mu\nu} \equiv \varepsilon^{\mu\nu\rho\sigma} \frac{\delta S}{\delta F^{\rho\sigma}} \,, \tag{2.15}$$

to formulate the duality transformations. The introduction of the tensor $G_{\mu\nu}$ also allows for a convenient way of computing the electric charge q and magnetic charge p, based on Stokes' theorem:

$$\begin{pmatrix} p \\ q \end{pmatrix} = -\frac{1}{2} \int_{\Sigma^2} \begin{pmatrix} F_{\mu\nu} \\ G_{\mu\nu} \end{pmatrix} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \,, \tag{2.16}$$

with Σ^2 the boundary of a volume containing the charges. A field configuration with only non-vanishing component $F_{\theta\varphi} = -p \sin\theta/(4\pi)$ satisfies Maxwell's equations with magnetic charge p, whereas a field configuration with only non-vanishing component $F_{rt} = q/(4\pi r^2)$ is a solution of Maxwell's equations, with electric charge q. Such field configurations will appear in our discussion of the dilaton black hole.

The resulting symmetry group is then $\mathrm{SL}(2,\mathbb{R})$ which also acts on the complex scalar field and on the charges (see for example Exercise 22.25). In the more general case, where m gauge fields are coupled to each other, it turns out that the duality transformations (in $D = 4$) form the symplectic group $\mathrm{Sp}(2m,\mathbb{R})$. We note that $\mathrm{Sp}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})$. We will encounter these groups later on.

3 The dilaton black hole

We first introduce the attractor mechanism in a simpler setting such that the main features are clear, before turning our attention to the more general case. Our model lives in $\mathcal{N} = 2$ supergravity and contains a gravity multiplet and a single gauge vector multiplet. This means that the bosonic fields are the metric $g_{\mu\nu}$, the graviphoton, the gauge multiplet photon and a complex scalar z, and we have two field strengths $F_{\mu\nu}$ and $F'_{\mu\nu}$. The target space for the scalars is the Poincaré plane, such that $\mathrm{Im}(z) > 0$.

In our example of a black hole attractor [3], it turns out that $\mathrm{Re}(z)$ vanishes[1] and hence we have only one real scalar $\phi(r)$, called the *dilaton*, which is related to z via $z = ie^{-2\phi}$. Hence the bosonic part of the action can be written as

$$S_b = \frac{1}{2\kappa^2}\int \mathrm{d}^4x\sqrt{-g}\left[R - 2\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}e^{-2\phi}\left(F^{\mu\nu}F_{\mu\nu} + F'^{\mu\nu}F'_{\mu\nu}\right)\right], \tag{3.1}$$

with $\kappa^2 = 8\pi G$. In isotropic coordinates, the line element can be written as

$$\mathrm{d}s^2 = -e^{2U(r)}\,\mathrm{d}t^2 + e^{-2U(r)}\left(\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right)\right). \tag{3.2}$$

In the remainder of this section, we will study the properties of this solution and see how the attractor phenomenon emerges from it.

3.1 The gauge fields

Let us start by considering the gauge fields in more detail. One of the gauge fields is purely electric, while the other is magnetic such that its dual is electric. Spherical symmetry then uniquely determines these 2-forms to be given by

$$F = \pm\,\mathrm{d}\left(\frac{1}{H_1}\right)\wedge\mathrm{d}t\,,\quad G' = \pm\,\mathrm{d}\left(\frac{1}{H_2}\right)\wedge\mathrm{d}t\,, \tag{3.3}$$

where H_1 and H_2 depend only on r, and G' is defined by

$$G'_{\mu\nu} \equiv -\tfrac{1}{2}\sqrt{-g}e^{-2\phi}\varepsilon_{\mu\nu\rho\sigma}F'^{\rho\sigma} = \kappa^2\varepsilon_{\mu\nu\rho\sigma}\frac{\delta S}{\delta F_{\rho\sigma}}\,, \tag{3.4}$$

[1] More general solutions, where $\mathrm{Re}(z) \neq 0$, are called *axion-dilaton* black holes [4].

which is equation (2.15), up to a factor κ^2. We redefine several quantities compared to other chapters of the *Supergravity* book by renaming $\mathcal{N}_{IJ} \to \kappa^{-2}\mathcal{N}_{IJ}$ (see Section 5.1.1), hence the factor κ^2 appearing. This means that electric charge becomes $q_{\text{new}} = \kappa^2 q_{\text{old}}$, and similar for magnetic charge. Recall that in $D = 4$, we have

$$ {}^*F_{\mu\nu} = \tfrac{1}{2}\sqrt{-g}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} \,, \tag{3.5} $$

such that $G'_{\mu\nu} = -e^{-2\phi}\,({}^*F')_{\mu\nu}$. Given the relations in Section 2.2, we can think of G' as the dual of F'. The full solution is given by

$$ e^{-2U} = H_1 H_2 \,, \qquad e^{-2\phi} = H_1/H_2 \tag{3.6} $$

$$ H_1 = e^{-\phi_0} + \frac{|q|}{4\pi r} \,, \quad H_2 = e^{\phi_0} + \frac{|p'|}{4\pi r} \,, \tag{3.7} $$

where we have defined ϕ_0 as the value of the dilaton field ϕ at spatial infinity. We do not prove explicitly that this solution solves all field equations.

There are four different possibilities for the signs of the charges, as suggested by the $\pm$ symbol in equation (3.3). We now check this with the following exercise, where we compute the magnetic 2-forms and evaluate their expressions at infinity.

Exercise 22.20

As a first intermediate step, we explicitly write out the components of F and G'. Note that

$$ \mathrm{d}\left(\frac{1}{H_1}\right) = -\frac{1}{H_1^2}\,\mathrm{d}H_1 = \frac{1}{H_1^2}\frac{|q|}{4\pi r^2}\,\mathrm{d}r \,, \tag{3.8} $$

and similarly, we have

$$ \mathrm{d}\left(\frac{1}{H_2}\right) = \frac{1}{H_2^2}\frac{|p'|}{4\pi r^2}\,\mathrm{d}r \,. \tag{3.9} $$

Hence we have that

$$ F = \pm\frac{1}{H_1^2}\frac{|q|}{4\pi r^2}\,\mathrm{d}r \wedge \mathrm{d}t \equiv \frac{1}{H_1^2}\frac{q}{4\pi r^2}\,\mathrm{d}r \wedge \mathrm{d}t \,, \tag{3.10} $$

$$ G' = \pm\frac{1}{H_2^2}\frac{|p'|}{4\pi r^2}\,\mathrm{d}r \wedge \mathrm{d}t \equiv -\frac{1}{H_2^2}\frac{p'}{4\pi r^2}\,\mathrm{d}r \wedge \mathrm{d}t \,, \tag{3.11} $$

where we defined $q \equiv \pm|q|$ and $p' \equiv \mp|p'|$. Only $F_{rt} = -F_{tr}$ are non-zero, and similarly for G'. These components are

$$ F_{rt} = \frac{1}{H_1^2}\frac{q}{4\pi r^2} \,, \quad G'_{rt} = \frac{1}{H_2^2}\frac{p'}{4\pi r^2} \,. \tag{3.12} $$

Let us now compute the 'dual' of F, which we call G and is calculated via equation (3.4). Because of the Levi-Civita symbol, and the fact that only the rt- and tr-component of F are non-zero, we see that the only non-zero components of G are $G_{\theta\varphi} = -G_{\varphi\theta}$, given by

$$ G_{\theta\varphi} = -\tfrac{1}{2}\sqrt{-g}e^{-2\phi}\varepsilon_{\theta\varphi\rho\sigma}F^{\rho\sigma} \,. \tag{3.13} $$

To compute them, note that the summation over ρ and σ gives twice the same result, due to antisymmetry of $F^{\rho\sigma}$ and the Levi-Civita symbol. For F^{rt}, we simply use the metric to raise the indices, and find

$$F^{rt} = g^{r\alpha} g^{t\beta} F_{\alpha\beta} = g^{rr} g^{tt} F_{rt} = -F_{rt}\,, \tag{3.14}$$

where we made use of our metric, as given by equation (3.2). The minus sign is cancelled by a sign from the Levi-Civita tensor. Finally, we need that $\sqrt{-g} = r^2 \sin\theta e^{-2U}$. Putting all results together, we find

$$G_{\theta\varphi} = -\frac{q}{4\pi}\sin\theta \frac{1}{H_1^2} e^{-2U} e^{-2\phi}\,. \tag{3.15}$$

We can get rid off the final combination of factors after we evaluate this expression at spatial infinity. Then $\phi = \phi_0$ by definition, and $H_1 = e^{-\phi_0}$ and $H_2 = e^{\phi_0}$ according to equations (3.7) such that $H_1 H_2 = 1$. Using equations (3.6), it follows that $e^{-2U} = 1$, and we are left with $e^{-2\phi_0}/H_1^2$, which is unity as well. Hence we find

$$G = -\frac{q}{4\pi}\sin\theta \,\mathrm{d}\theta \wedge \mathrm{d}\varphi\,. \tag{3.16}$$

Let us now look at the calculation of F'. We start by showing that

$$F'_{\mu\nu} = \tfrac{1}{2}\sqrt{-g} e^{2\phi} \varepsilon_{\mu\nu\rho\sigma} G'^{\rho\sigma}\,. \tag{3.17}$$

To see this, let us reconsider the definition of the components of G' as function of the components of F', see equation (3.4). Using the definitions of the dual and tilde operation from Section 2.2, we find that

$$G'_{\mu\nu} = -e^{-2\phi}\left({}^* F'_{\mu\nu}\right) = -i e^{-2\phi} \tilde{F}'_{\mu\nu}\,, \tag{3.18}$$

where we absorbed the factor κ^2. Rearranging factors, and applying the tilde operation on both sides of the equation, we find

$$F'_{\mu\nu} = i e^{2\phi} \tilde{G}'_{\mu\nu} = e^{2\phi}\left({}^* G'_{\mu\nu}\right)\,, \tag{3.19}$$

where we made use of the fact that tilde squares to the identity. Using equation (3.5), this can then be written as

$$F'_{\mu\nu} = \tfrac{1}{2}\sqrt{-g} e^{2\phi} \varepsilon_{\mu\nu\rho\sigma} G'^{\rho\sigma}\,. \tag{3.20}$$

Now, the steps are essentially identical to the derivation of G. We end up with

$$F'_{\theta\varphi} = -\frac{p'}{4\pi}\sin\theta \frac{1}{H_2^2} e^{-2U} e^{2\phi} \tag{3.21}$$

as the only non-zero components of F'. Again, the combination of factors at the end becomes unity if we make use of equations (3.6), such that

$$F' = -\frac{p'}{4\pi}\sin\theta \,\mathrm{d}\theta \wedge \mathrm{d}\varphi\,. \tag{3.22}$$

3.2 Example of attractor mechanism

We are now ready to illuminate the attractor mechanism in the dilaton black hole solution. First, we show that the mass of the black hole depends on the value of the dilaton at spatial infinity, ϕ_0. Let us look at the g_{rr} component, which can be expanded using our definitions of $H_{1,2}$ to find

$$-g^{tt} = g_{rr} = 1 + \frac{e^{-\phi_0}|p'| + e^{\phi_0}|q|}{4\pi r} + \frac{|qp'|}{(4\pi r)^2}\,. \tag{3.23}$$

The horizon of the black hole is located at $r = 0$. We can compare with the g_{rr} component from the Reissner-Nördstrom metric: see equation (2.6). We see that the mass of the black hole is determined by the coefficient of the $1/r$ term in g_{rr}, and hence find

$$8\pi MG = e^{-\phi_0}|p'| + e^{\phi_0}|q|\,, \tag{3.24}$$

such that the mass directly depends on $|p'|, |q|$ and ϕ_0. However, the value of the dilaton field at the horizon depends *only* on the charges. By expanding H_1/H_2, we find

$$e^{-\phi} = \frac{|q| + 4\pi r e^{-\phi_0}}{|p'| + 4\pi r e^{\phi_0}}\,. \tag{3.25}$$

This tells us that the value of $e^{-2\phi}$ at the horizon $r = 0$ is given by

$$(e^{-2\phi})|_{r=0} \equiv (e^{-2\phi_h}) = \left|\frac{q}{p'}\right|\,, \tag{3.26}$$

which is independent of the value of the dilaton field at spatial infinity. For all choices of $\phi_0 = \phi(r = +\infty)$, the value of $\phi(r = 0)$ is fixed by the properties of the black hole. All possible boundary conditions for ϕ have hence the same value at the horizon: this is precisely the attractor mechanism that we anticipated earlier on.

3.3 Horizon area

Like the extremal Reissner-Nördstrom black hole, the metric of the dilaton black hole approaches the Robinson-Bertotti metric near the horizon. Hence the dilaton black hole can be seen as interpolating between two vacua: a flat space at spatial infinity, and a Robinson-Bertotti space at $r = 0$. This interpretation comes from the gradient flow behaviour that we will discuss in Section 6. We now compute the area of the horizon, show that this indeed agrees with the Robinson-Bertotti metric, and also compute the invariants $F_{\mu\nu}F^{\mu\nu}$, $F'_{\mu\nu}F'^{\mu\nu}$ at the horizon.

Exercise 22.21

For convenience we look at a specific timeslice $t = T =$ cte which does not alter our conclusions since the metric is static. In this case, it is clear that the horizon is a 2-sphere with radius r, parametrized by the angles θ and φ. To compute the area, we therefore compute the area of a 2-sphere with radius R, and consider the limit $R \to 0$ to obtain the area of the horizon.

A 2-sphere with radius R is a 2-dimensional submanifold embedded in our space-time. The line element on this submanifold can be found by plugging $t = T$, $r = R$ into our original line element, given in equation (3.2). This gives

$$\mathrm{d}s^2 = e^{-2U} R^2 \left(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\varphi^2\right) . \tag{3.27}$$

The induced metric $h_{\alpha\beta}$ can be read off from this line element. The area $A(R)$ of this submanifold is computed via the formula

$$A(R) = \int \sqrt{h}\, \mathrm{d}^{\mathrm{k}}\xi \,, \tag{3.28}$$

where h is the determinant of the metric, and the coordinates parametrizing our submanifold are $\xi^1 = \theta$, $\xi^2 = \varphi$. Hence we find

$$A(R) = e^{-2U} R^2 \iint \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\varphi = 4\pi e^{-2U} R^2 . \tag{3.29}$$

The area of the horizon can be found by taking the limit of this expression for $R \to 0$. For this, we can use that $e^{-2U} = H_1 H_2$ as defined earlier. We then find

$$A = \lim_{R\to 0} A(R) = 4\pi \lim_{R\to 0} \left[R^2 \left(e^{-\phi} + \frac{|q|}{4\pi R} \right) \left(e^{\phi} + \frac{|p'|}{4\pi R} \right) \right] = \frac{|qp'|}{4\pi} . \tag{3.30}$$

For later convenience, we repeat here explicitly that the previous calculation essentially amounts to

$$\lim_{r\to 0} \left(r^2 e^{-2U} \right) = \frac{|qp'|}{(4\pi)^2} . \tag{3.31}$$

If we use a similar reasoning to compute the horizon area for the Robinson-Bertotti metric, we find

$$A = 4\pi (MG)^2 . \tag{3.32}$$

This agrees with the above result: we can get $(MG)^2$ by comparing the $1/r^2$ terms of the g_{rr} metric component between the dilaton black hole and the Reissner-Nördstrom metric. This shows that $(MG)^2 = |qp'|/(4\pi)^2$, and the two areas agree.

We now check that the invariants $F^{\mu\nu}F_{\mu\nu}$ and $F'^{\mu\nu}F'_{\mu\nu}$ are non-singular and constant on the horizon. Let us first focus on the former. From our calculations of Exercise 22.20, we find that

$$F_{\mu\nu}F^{\mu\nu} = 2F_{rt}F^{rt} = -2\left(F_{rt}\right)^2 = -2\frac{1}{H_1^4}\frac{q^2}{(4\pi)^2 r^4} \,, \tag{3.33}$$

where we used equation (3.12) for F_{rt}. This quantity is non-singular on the horizon, since

$$\lim_{r\to 0} \left(r H_1 \right) = \lim_{r\to 0} \left[r \left(e^{-\phi} + \frac{|q|}{4\pi r} \right) \right] = \frac{|q|}{4\pi} . \tag{3.34}$$

Therefore, the invariant on the horizon evaluates to

$$\left(F_{\mu\nu}F^{\mu\nu}\right)\Big|_{r=0} = -\frac{32\pi^2}{q^2} . \tag{3.35}$$

To show that the other invariant $F'^{\mu\nu}F'_{\mu\nu}$ is non-singular and constant, we note that

$$F'^{\mu\nu}F'_{\mu\nu} = 2F'^{\theta\varphi}F'_{\theta\varphi} = 2g^{\theta\theta}g^{\varphi\varphi}(F'_{\theta\varphi})^2$$
$$= 2\frac{p'^2}{(4\pi)^2}e^{4U}r^{-4}\,, \tag{3.36}$$

where we used results from Exercise 22.20 for $F'_{\theta\varphi}$. From our result in equation (3.31), we then find that on the horizon, the above evaluates to

$$\left(F'^{\mu\nu}F'_{\mu\nu}\right)\big|_{r=0} = \frac{32\pi^2}{q^2}\,. \tag{3.37}$$

4 General black hole attractors

We will now develop a more general[2] treatment of the black hole attractor mechanism. In this section, we prepare ourselves by discussing the metric ansatz and its relation with non-extremal black holes. Then, we will generalize our results for the field strengths to include arbitrary charges (p,q), (p',q'). The dilaton black hole can be obtained from the general solution by setting $p=0$, $q'=0$ and $z=ie^{-2\phi}$. Next, we discuss the equations of motion for extremal black holes and how to solve them. After discussing supersymmetric solutions and central charges in the next section, we will be able to reveal the attractor mechanism of these more general black hole solutions, as well as prove that they are supersymmetric solutions of the theory.

4.1 Metric ansatz

We introduce a new radial coordinate $\tau = r^{-1}$, such that our metric ansatz now reads

$$\mathrm{d}s^2 = -e^{2U(\tau)}\,\mathrm{d}t^2 + e^{-2U(\tau)}\left[\frac{\mathrm{d}\tau^2}{\tau^4} + \frac{1}{\tau^2}\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right)\right]. \tag{4.1}$$

We note that the Levi-Civita tensor has convention $\varepsilon_{t\tau\theta\varphi} = -1$, in order to agree with our earlier convention. This line element is the limit of $c \to 0$ of a more general line element:

$$\mathrm{d}s^2 = -e^{2U(\tau)}\,\mathrm{d}t^2 + e^{-2U(\tau)}\left[\frac{c^4\,\mathrm{d}\tau^2}{\sinh^4(c\tau)} + \frac{c^2}{\sinh^2(c\tau)}\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right)\right]. \tag{4.2}$$

The idea is that for any choice of charges (p,q) and (p',q'), we have a whole *family* of black hole solutions described by the line element in equation (4.2) with parameter c. For $c \to 0$, the solutions become extremal black holes. The parameter c is therefore called the *non-extremality parameter*. To further support this claim, we now show that the metric (4.2) is equivalent with the Reissner-Nördstrom metric, if $c = (r_+ - r_-)/2$. Then $c = 0$ indeed corresponds to the extremal case where the

[2] The action can be found from equation (7.1) with $n_V = 1$ and using the information given in equation (5.10).

Reissner-Nördstrom black hole has a single horizon. However, this text will only discuss extremal solutions later on.

Exercise 22.22

Recall that the non-extremal Reissner-Nördstrom metric is

$$\mathrm{d}s^2 = -F(r)\,\mathrm{d}t^2 + \frac{1}{F(r)}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega_2^2\,, \tag{4.3}$$

with

$$F(r) = 1 - \frac{2MG}{r} + \frac{q^2 G}{4\pi r^2} = \frac{(r-r_+)(r-r_-)}{r^2}\,. \tag{4.4}$$

The change of variable leading from one metric to the other is

$$\frac{c^2}{\sinh^2(c\tau)} = (r-r_+)(r-r_-)\,. \tag{4.5}$$

From comparing the two metrics and given the fact that the time coordinate remains the same, we see that we have to define $U(\tau)$ via

$$e^{2U(\tau)} = F(r)\,. \tag{4.6}$$

This automatically ensures that the metric of the 2-sphere agrees as well. Indeed, we have the equality

$$e^{-2U(\tau)}\frac{c^2}{\sinh^2(c\tau)} = r^2\,, \tag{4.7}$$

if we combine equations (4.5) and (4.6). Comparing the radial parts, we find that the final match between the metrics requires

$$e^{-2U(\tau)}\frac{c^4}{\sinh^4(c\tau)} = \frac{1}{F(r)}\,\mathrm{d}r^2\,, \tag{4.8}$$

which, upon using equation (4.6), can be written as

$$\frac{c^4}{\sinh^4(c\tau)} = \mathrm{d}r^2\,. \tag{4.9}$$

We will now show that this final relation implies that $c = (r_+ - r_-)/2$. For this, we start from equation (4.5), and differentiate both sides to find

$$-\,c^3\sinh^{-3}(c\tau)\cosh(c\tau)\,\mathrm{d}\tau = \left(r - \frac{(r_+ + r_-)}{2}\right)\mathrm{d}r\,. \tag{4.10}$$

We square this equation and find

$$c^6\sinh^{-6}(c\tau)\cosh^2(c\tau)\,\mathrm{d}\tau^2 = \left(r - \frac{(r_+ + r_-)}{2}\right)^2\mathrm{d}r^2\,. \tag{4.11}$$

Now we make use of equation (4.9) to find

$$c^2 \sinh^{-2}(c\tau)\cosh^2(c\tau) = \left(r - \frac{(r_+ + r_-)}{2}\right)^2 . \tag{4.12}$$

Recall the well-known identity $\cosh^2 x - \sinh^2 x = 1$, such that

$$c^2 = \left(r - \frac{r_+ + r_-}{2}\right)^2 - (r - r_+)(r - r_-)\,, \tag{4.13}$$

where we again made use of equation (4.5). Expanding the squares, we find that the right hand side evaluates to a constant:

$$c^2 = \left(\frac{r_+ + r_-}{2}\right)^2 - r_+ r_-\,, \tag{4.14}$$

from which we deduce that indeed $c = (r_+ - r_-)/2$, as was to be shown.

For applications involving more general field strengths, including the generalisation to multiple vector fields in the theory, it will prove convenient to show a few results valid for our metric ansatz from equation (4.1).

Exercise 22.23

Suppose that $F_{\mu\nu}$ is an antisymmetric tensor. We now show that

$$\tilde{F}_{t\tau} = i\frac{e^{2U}}{\sin\theta}F_{\theta\varphi}\,, \quad \tilde{F}_{\theta\varphi} = -ie^{-2U}\sin\theta F_{t\tau}\,. \tag{4.15}$$

First, recall that we have the relation

$$\tilde{F}_{\mu\nu} = -i\left({}^*F\right)_{\mu\nu}\,, \tag{4.16}$$

and we can again use the property from equation (3.5) to find

$$\tilde{F}_{\mu\nu} = -\tfrac{i}{2}\sqrt{-g}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}\,. \tag{4.17}$$

The metric that we are considering right now has

$$g_{\mu\nu} = \mathrm{diag}\left(-e^{2U}, e^{-2U}\tau^{-4}, e^{-2U}\tau^{-2}, e^{-2U}\tau^{-2}\sin^2\theta\right)\,, \quad \sqrt{-g} = e^{-2u}\tau^{-4}\sin\theta\,. \tag{4.18}$$

Now observe that

$$\tilde{F}_{t\tau} = i\sqrt{-g}g^{\theta\theta}g^{\varphi\varphi}F_{\theta\phi} = i\frac{e^{-2U}}{\sin\theta}F_{\theta\varphi}\,, \tag{4.19}$$

where we used that $\varepsilon_{t\tau\theta\varphi} = -1$ and, as before, the fact that the summation over $\rho\sigma$ gives a factor 2 due to antisymmetry of the two tensors being contracted. Similarly, we find

$$\tilde{F}_{\theta\varphi} = i\sqrt{-g}g^{tt}g^{\tau\tau}F_{t\tau} = -ie^{-2U}\sin\theta F_{t\tau}\,, \tag{4.20}$$

where the sign flip is due to the g^{tt} factor.

We now prove some useful results for the multi-component case, for which the Lagrangian is given in equation (21.4) of the book (again up to factors κ^2). As before, we define a 2-form $G_{\mu\nu I}$ for each 2-form $F^I_{\mu\nu}$ as

$$G_{\mu\nu I} \equiv \kappa^2 \varepsilon_{\mu\nu\rho\sigma} \frac{\delta S}{\delta F^I_{\rho\sigma}} = i I_{IJ} \tilde{F}^J_{\mu\nu} + R_{IJ} F^J_{\mu\nu}\,, \tag{4.21}$$

where $I \equiv \mathrm{Im}(\mathcal{N}_{IJ})$ and $R \equiv \mathrm{Re}(\mathcal{N}_{IJ})$ (see Section 5.1.1). The index I has values $0, \ldots, n_V$ (with n_V the total number of vector multiplets) and labels the different multiplets in the theory. Its index placement is important for the symplectic formulation, which is discussed later on. We write this equation explicitly for $\mu\nu = t\tau$ and $\theta\varphi$ and solve the equations. We obtain

$$G_{t\tau I} = i I_{IJ} \tilde{F}^J_{t\tau} + R_{IJ} F^J_{t\tau} = -\frac{e^{2U}}{\sin\theta} I_{IJ} F^J_{\theta\varphi} + R_{IJ} F^J_{t\tau}\,, \tag{4.22}$$

where we used the result for $\tilde{F}_{t\tau}$ we found above. Similarly, we can derive

$$G_{\theta\varphi I} = e^{-2U} \sin\theta I_{IJ} F^J_{t\tau} + R_{IJ} F^J_{\theta\varphi}\,. \tag{4.23}$$

We now prove we can neatly summarize this in the equation

$$\begin{pmatrix} F^I_{t\tau} \\ G_{t\tau I} \end{pmatrix} = -\Omega\mathcal{M} \frac{e^{2U}}{\sin\theta} \begin{pmatrix} F^I_{\theta\varphi} \\ G_{\theta\varphi I} \end{pmatrix}\,, \tag{4.24}$$

where Ω, the *symplectic metric*, and $\mathcal{M}$ are the matrices

$$\Omega = \begin{pmatrix} 0 & \delta_I{}^J \\ -\delta^I{}_J & 0 \end{pmatrix}\,, \quad \mathcal{M} = \begin{pmatrix} -(I + RI^{-1}R)_{JK} & (RI^{-1})_J{}^K \\ (I^{-1}R)^J{}_K & -(I^{-1})^{JK} \end{pmatrix}\,. \tag{4.25}$$

To show this, note that

$$\Omega\mathcal{M} = \begin{pmatrix} (I^{-1}R)^I{}_K & -(I^{-1})^{IK} \\ (I + RI^{-1}R)_{IK} & -(RI^{-1})_I{}^K \end{pmatrix}\,. \tag{4.26}$$

Then we have that

$$\Omega\mathcal{M} \begin{pmatrix} F^K_{\theta\varphi} \\ G_{\theta\varphi K} \end{pmatrix} = \begin{pmatrix} (I^{-1}R)^I{}_K F^K_{\theta\varphi} - (I^{-1})^{IK} G_{\theta\varphi K} \\ (I + RI^{-1}R)_{IK} F^K_{\theta\varphi} - (RI^{-1})_I{}^K G_{\theta\varphi K} \end{pmatrix}\,. \tag{4.27}$$

We will show this result is equal to $e^{-2U}\sin\theta\begin{pmatrix}F^I_{t\tau}\\ G_{t\tau I}\end{pmatrix}$, using earlier results. The first line of the matrix in equation (4.27) reads

$$\begin{aligned}(I^{-1}R)^I{}_K F^K_{\theta\varphi} - (I^{-1})^{IK} G_{\theta\varphi K} &= (I^{-1}R)^I{}_K F^K_{\theta\varphi} - (I^{-1})^{IK}\left(e^{-2U}\sin\theta I_{KL}F^L_{t\tau} + R_{KL}F^L_{\theta\varphi}\right)\\ &= (I^{-1}R)^I{}_K F^K_{\theta\varphi} - e^{-2U}\sin\theta F^I_{t\tau} - (I^{-1}R)^I{}_L F^L_{\theta\varphi}\\ &= -e^{-2U}\sin\theta F^I_{t\tau}\,,\end{aligned} \tag{4.28}$$

as was to be shown. The second entry can be manipulated as follows, again by substituting our result for $G_{\theta\varphi K}$:

$$\begin{aligned}&(I+RI^{-1}R)_{IK}F^K_{\theta\varphi} - (RI^{-1})_I{}^K\left(e^{-2U}\sin\theta I_{KL}F^L_{t\tau} + R_{KL}F^L_{\theta\varphi}\right)\\ &= I_{IK}F^K_{\theta\varphi} - e^{-2U}\sin\theta R_{IL}F^L_{t\tau}\\ &= -e^{-2U}\sin\theta\left(-\frac{e^{2U}}{\sin\theta}I_{IJ}F^J_{\theta\varphi} + R_{IJ}F^J_{t\tau}\right) = -e^{-2U}\sin\theta G_{t\tau I}\,,\end{aligned} \tag{4.29}$$

as was to be shown. From this result, we can also easily derive that

$$\frac{e^{2U}}{\sin\theta}\begin{pmatrix}F_{\theta\varphi}\\ G_{\theta\varphi}\end{pmatrix} = \Omega\mathcal{M}\begin{pmatrix}F_{t\tau}\\ G_{t\tau}\end{pmatrix} \tag{4.30}$$

without much effort. This is simply a consequence of the symplectic structure. Indeed, we will show in Exercise 22.28 that $\mathcal{M}$ is an element of $\mathrm{Sp}(2(n_V+1),\mathbb{R})$, which means that $\mathcal{M}\Omega\mathcal{M} = \Omega$. If $A = -\Omega\mathcal{M}B$, with A, B arbitrary symplectic vectors, then we can multiply on the left with $\mathcal{M}$, which gives $\mathcal{M}A = -\Omega B$. Now multiply on the left with Ω, to find $\Omega\mathcal{M}A = -\Omega^2 B$. The inverse of the symplectic metric is $\Omega^{-1} = -\Omega$, such that we end up with $B = \Omega\mathcal{M}A$. This reasoning is precisely what brings us to equation (4.30) if we start from equation (4.24).

As already mentioned in the previous exercise, the matrix $\mathcal{M}$ is an element of the symplectic group, in agreement with our recap from Section 2.2. We now prove this explicitly.

Exercise 22.28

To show that the matrix $\mathcal{M}$ is an element of $\mathrm{Sp}(2(n_V+1),\mathbb{R})$, we have to prove that $\mathcal{M}\Omega\mathcal{M} = \Omega$. We use equation (4.26) for the result of $\Omega\mathcal{M}$, and find

$$\mathcal{M}\Omega\mathcal{M} = \begin{pmatrix}-(I+RI^{-1}R)_{JK} & (RI^{-1})_J{}^K\\ (I^{-1}R)^J{}_K & -(I^{-1})^{JK}\end{pmatrix}\begin{pmatrix}(I^{-1}R)^K{}_L & -(I^{-1})^{KL}\\ (I+RI^{-1}R)_{KL} & -(RI^{-1})_K{}^L\end{pmatrix} \tag{4.31}$$

Investigating each entry of the resulting matrix separately shows that this indeed results in Ω. For example, the 11-component yields

$$-(I+RI^{-1}R)_{JK}(I^{-1}R)^K{}_L + (RI^{-1})_J{}^K(I+RI^{-1}R)_{KL} = \mathbf{0}_{JL}\,, \tag{4.32}$$

and the 12-entry becomes

$$(I + RI^{-1}R)_{JK}(I^{-1})^{KL} - (RI^{-1})_J{}^K(RI^{-1})_K{}^L = \delta_J{}^L\,, \tag{4.33}$$

and similarly for the 21- and 22-component of the matrix.

In case we have only one complex scalar z, we can use the results from Exercise 20.18 and Exercise 20.19, which are given in equation (5.10) in this text. Then the matrix $\mathcal{M}$ becomes

$$\mathcal{M} = \frac{1}{\mathrm{Im}(z)}\begin{pmatrix} |z|^2 & \mathrm{Re}(z) \\ \mathrm{Re}(z) & 1 \end{pmatrix}\,. \tag{4.34}$$

4.2 Field equations and black hole potential

Here we briefly discuss some useful tricks for solving the field equations. We will not derive expressions in detail and not show explicitly that our results indeed solve the field equations, but rather focus on conceptual aspects. In particular, solving the Einstein equations is simplified by introducing an auxiliary function called the *black hole potential*. Apart from its aid in solving the field equations, the black hole potential turns out to be intimately linked with the central charge, to be introduced in Section 5.2, and the area of the black hole horizon. It will prove to be an important tool to show the attractor behaviour of the general solutions in Section 6.

4.2.1 Gauge fields

To solve for the gauge fields $F_{\mu\nu}$ and $F'_{\mu\nu}$, we start from the assumption that they describe dyons with charges (p,q) and (p',q'), respectively. Using spherical symmetry and the Bianchi identity severely restricts their expressions such that only $F_{t\tau}$ and $F_{\theta\varphi}$ are non-zero. The exact expressions are determined by varying the action with respect to the gauge fields and integrating, where the charges p, p' and q, q' appear as integration constant. The result can neatly be summarized by displaying F and G as a symplectic vector, using the results from Exercise 22.23:

$$4\pi\begin{pmatrix} F \\ G \end{pmatrix} = e^{2U}\Omega\mathcal{M}\begin{pmatrix} p \\ q \end{pmatrix}\mathrm{d}t\wedge\mathrm{d}\tau - \begin{pmatrix} p \\ q \end{pmatrix}\sin\theta\,\mathrm{d}\theta\wedge\mathrm{d}\varphi\,, \tag{4.35}$$

where $\mathcal{M}$ is given by equation (4.34). The equations for F' and G' are found using the same formula and replacing $p \to p'$, $q \to q'$. It will prove convenient in later exercises to work out F and G individually by expanding the matrix product in the first term. Hence we find

$$F = \frac{e^{2U}}{4\pi}\frac{1}{\mathrm{Im}(z)}\left(\mathrm{Re}(z)p + q\right)\mathrm{d}t\wedge\mathrm{d}\tau - \frac{p}{4\pi}\sin\theta\,\mathrm{d}\theta\wedge\mathrm{d}\varphi \tag{4.36}$$

$$G = -\frac{e^{2U}}{4\pi}\frac{1}{\mathrm{Im}(z)}\left(|z|^2 p + \mathrm{Re}(z)q\right)\mathrm{d}t\wedge\mathrm{d}\tau - \frac{q}{4\pi}\sin\theta\,\mathrm{d}\theta\wedge\mathrm{d}\varphi\,. \tag{4.37}$$

We can also go back to our original radial coordinate r by using the change of variable $\tau = r^{-1}$, such that the above equations read

$$F = -\frac{e^{2U}}{4\pi r^2}\frac{1}{\mathrm{Im}(z)}\left(\mathrm{Re}(z)p + q\right)\mathrm{d}t\wedge\mathrm{d}r - \frac{p}{4\pi}\sin\theta\,\mathrm{d}\theta\wedge\mathrm{d}\varphi \tag{4.38}$$

$$G = \frac{e^{2U}}{4\pi r^2}\frac{1}{\mathrm{Im}(z)}\left(|z|^2 p + \mathrm{Re}(z)q\right)\mathrm{d}t\wedge\mathrm{d}r - \frac{q}{4\pi}\sin\theta\,\mathrm{d}\theta\wedge\mathrm{d}\varphi\,. \tag{4.39}$$

We can easily check that the field strengths of the dilaton black hole solution agree with this general form if we have charges $(0, q)$ and $(p', 0)$, and put $\mathrm{Re}(z) = 0$, $\mathrm{Im}(z) = e^{-2\phi}$.

Exercise 22.26

For F, the second term vanishes since $p = 0$. Hence we have

$$F = -\frac{e^{2U}e^{2\phi}}{4\pi r^2}q\,\mathrm{d}t\wedge\mathrm{d}r\,. \tag{4.40}$$

By making use of the result that $e^{-2U} = H_1H_2$ and $e^{-2\phi} = H_1/H_2$ and swapping the differentials (resulting in a minus sign), we indeed find the result from equation (3.10). For F', we use the same equation but with primes on the charges. Now we have $q' = 0$ and since $\mathrm{Re}(z) = 0$, the first term vanishes. Hence the second term remains and indeed agrees with the result we found in Exercise 22.20. For G, again the first term vanishes since $\mathrm{Re}(z) = 0$ and $p = 0$, and the second term then agrees with Exercise 22.20. For G', the second term vanishes since $q' = 0$, and the first term reads

$$G' = \frac{e^{2U}e^{2\phi}}{4\pi r^2}|z|^2 p'\,\mathrm{d}t\wedge\mathrm{d}r\,. \tag{4.41}$$

Now it only remains to realize that $\mathrm{Re}(z) = 0$ implies that $|z| = \mathrm{Im}(z) = e^{-2\phi}$, such that the numerator has a factor $e^{2U}e^{-2\phi} = 1/H_2^2$. Then we indeed find the same result as equation (3.11) if we again swap the differentials.

4.2.2 Einstein equations and black hole potential

One also needs to solve the Einstein equations

$$R_{\mu\nu} = \kappa^2\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^\rho_\rho\right)\,, \tag{4.42}$$

with energy-momentum tensor

$$\kappa^2 T_{\mu\nu} = \mathrm{Im}(z)\left(F_{\mu\rho}F_\nu{}^\rho - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} + (F\to F')\right) + \frac{1}{4(\mathrm{Im}(z))^2}\left(\partial_\mu z\partial_\nu\bar{z} + \partial_\nu z\partial_\mu\bar{z} - g_{\mu\nu}\partial_\rho z\partial^\rho\bar{z}\right)\,, \tag{4.43}$$

where the notation $F \to F'$ tells us we should repeat the first two terms but replace F by F'. The only non-zero components are the diagonal ones. We can simplify the equations by introducing the *black hole potential*:

$$(4\pi)^2 V_{\rm BH} = \frac{1}{2}\begin{pmatrix} p & q \end{pmatrix} \mathcal{M} \begin{pmatrix} p \\ q \end{pmatrix} + \frac{1}{2}\begin{pmatrix} p' & q' \end{pmatrix} \mathcal{M} \begin{pmatrix} p' \\ q' \end{pmatrix} . \tag{4.44}$$

$$\equiv (4\pi)^2 V_{\rm BH}^{(p,q)} + (4\pi)^2 V_{\rm BH}^{(p',q')} . \tag{4.45}$$

We will dedicate some time on the black hole potential, since it will play a crucial role in deriving the gradient flow equations in Section 6. The following exercise shows that the black hole potential is always positive.

Exercise 22.24

It is sufficient to show that each term individually on the right hand side of equation (4.44) is positive, such that their sum is positive as well. To show this, note that the matrix product gives

$$\frac{1}{2}\begin{pmatrix} p & q \end{pmatrix} \mathcal{M} \begin{pmatrix} p \\ q \end{pmatrix} = \frac{1}{2\,{\rm Im}(z)} \left(|z|^2 p^2 + 2\,{\rm Re}(z) pq + q^2\right) . \tag{4.46}$$

We get a lower bound on this potential by using the basic fact that ${\rm Re}(z) \geq -|z|$. Then the above equation tells us that

$$(4\pi)^2 V_{\rm BH}^{(p,q)} \geq \frac{1}{2\,{\rm Im}(z)} \left(|z|^2 p^2 - 2|z| pq + q^2\right) = \frac{1}{2\,{\rm Im}(z)} \left(|z|p - q\right)^2 \geq 0 \, , \tag{4.47}$$

since by assumption ${\rm Im}(z) > 0$. This shows the black hole potential is indeed positive. We can even say more, and relate $V_{\rm BH}$ to the energy density which we obtain by inserting the gauge fields F and F', as given by equation (4.36), in the electromagnetic terms of the stress tensor in equation (4.43). In the intermediate steps below, we ignore the factor κ^{-2} in front of the energy-momentum tensor. The term involving F in the T_{00} component is

$$T_{00} = {\rm Im}(z) \left(F_{t\tau} F_t{}^{\tau} - \frac{1}{2} g_{tt} \left(F_{t\tau} F^{t\tau} + F_{\theta\varphi} F^{\theta\varphi} \right) \right) + \dots \tag{4.48}$$

$$= {\rm Im}(z) \left(g^{\tau\tau} (F_{t\tau})^2 - \frac{1}{2} g_{tt} \left(g^{tt} g^{\tau\tau} (F_{t\tau})^2 + g^{\theta\theta} g^{\varphi\varphi} (F_{\theta\varphi})^2 \right) \right) + \dots \tag{4.49}$$

$$= \frac{{\rm Im}(z)}{2} \left(g^{\tau\tau} (F_{t\tau})^2 - g_{tt} g^{\theta\theta} g^{\varphi\varphi} (F_{\theta\varphi})^2 \right) + \dots \, , \tag{4.50}$$

where the dots each time denote the terms involving F' (the terms involving the scalar are not considered here). Note that we heavily made use of the fact that our metric is diagonal. From the result in equation (4.36), we gather that

$$F_{t\tau} = \frac{e^{2U}}{4\pi\,{\rm Im}(z)} \left({\rm Re}(z) p + q\right) \, , \quad F_{\theta\varphi} = -\frac{p}{4\pi} \sin\theta \, . \tag{4.51}$$

Using this along with the metric we are considering, we find

$$T_{00} = \frac{\mathrm{Im}(z)}{2(4\pi)^2} e^{6U} \tau^4 \left(\frac{1}{(\mathrm{Im}(z))^2} \left((\mathrm{Re}(z))^2 p^2 + 2\,\mathrm{Re}(z) pq + q^2 \right) + p^2 \right) + \dots \tag{4.52}$$

$$= \frac{1}{(4\pi)^2\,\mathrm{Im}(z)} e^{6U} \tau^4 \left[\tfrac{1}{2} \left(|z|^2 p^2 + 2\,\mathrm{Re}(z) pq + q^2 \right) \right] + \dots \tag{4.53}$$

$$= e^{6U} \tau^4 V_{\mathrm{BH}}^{(p,q)} + \dots , \tag{4.54}$$

where $V_{\mathrm{BH}}^{(p,q)}$ was defined earlier. The term involving F' follows an identical derivation, so we conclude that

$$T_{00} = e^{6U} \tau^4 V_{\mathrm{BH}} , \tag{4.55}$$

when only considering the electromagnetic terms in the energy-momentum tensor. Therefore, since $\sqrt{-g} = \tau^{-4} e^{-2U(\tau)} \sin\theta$, and $g^{tt} = -e^{-2U}$, we find

$$\sqrt{-g} T^0{}_0 = \sqrt{-g} g^{tt} T_{00} = -\kappa^{-2} e^{2U} V_{\mathrm{BH}} \sin\theta , \tag{4.56}$$

where we restored the factor κ^{-2}. So we can conclude that, at least concerning the radial dependence, $e^{2U} V_{\mathrm{BH}}$ is essentially the electromagnetic energy density $\sqrt{-g} T^0{}_0$.

A convenient property of the black hole potential is the fact that it is invariant under action of the duality transformations, *i.e.* the group $\mathrm{SL}(2,\mathbb{R})$ in this context. We introduce notation to show this in the following exercise: the scalar z and the charges (p,q) and (p',q') transform under the action of $\mathrm{SL}(2,\mathbb{R})$ as

$$z \to \frac{az - b}{d - cz} , \qquad \begin{pmatrix} p \\ q \end{pmatrix} \to \begin{pmatrix} d & c \\ b & a \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} , \tag{4.57}$$

and similarly for the charges (p',q').

Exercise 22.25

To show this, it is sufficient to prove it for the generators of the group, which are (i) inversion ($a = d = 0$, $b = -c = 1$), (ii) translation ($a = d = 1$, $c = 0$, $b \in \mathbb{R}$) and (iii) scale transformations ($d = 1/a$, $b = c = 0$). Note that it is again sufficient to consider one of the terms on the right hand side of equation (4.44), since the transformation does not mix the charges (p,q) with (p',q'). Also recall from the previous exercise that

$$(4\pi)^2 V_{\mathrm{BH}}^{(p,q)} = \frac{1}{2\,\mathrm{Im}(z)} \begin{pmatrix} p & q \end{pmatrix} \mathcal{M} \begin{pmatrix} p \\ q \end{pmatrix} = \frac{1}{2} \left(|z|^2 p^2 + 2\,\mathrm{Re}(z) pq + q^2 \right) . \tag{4.58}$$

(i) Inversion is the transformation

$$z \to \frac{-1}{z} , \qquad \begin{pmatrix} p \\ q \end{pmatrix} \to \begin{pmatrix} -q \\ p \end{pmatrix} . \tag{4.59}$$

It is fairly easy to show that

$$\mathrm{Im}\left(-\frac{1}{z}\right) = \frac{\mathrm{Im}(z)}{|z|^2}\,, \quad \mathrm{Re}\left(-\frac{1}{z}\right) = -\frac{\mathrm{Re}(z)}{|z|^2}\,, \tag{4.60}$$

such that

$$V_{\mathrm{BH}}^{(p,q)} \to \frac{|z|^2}{\mathrm{Im}(z)}\left(|z|^{-2}q^2 + |z|^{-2}\,\mathrm{Re}(z)pq + p^2\right) = V_{\mathrm{BH}}^{(p,q)}\,, \tag{4.61}$$

which shows that the black hole potential is invariant under inversions.
(ii) Translations act as

$$z \to z - b\,, \begin{pmatrix} p \\ q \end{pmatrix} \to \begin{pmatrix} p \\ bp + q \end{pmatrix}\,. \tag{4.62}$$

Note that then $\mathrm{Re}(z) \to \mathrm{Re}(z) - b$ and $|z|^2 \to |z|^2 - 2b\,\mathrm{Re}(z) + b^2$, such that

$$V_{\mathrm{BH}}^{(p,q)} \to \frac{1}{2\,\mathrm{Im}(z)}\left(\left(|z|^2 - 2b\,\mathrm{Re}(z) + b^2\right)p^2 + 2(\mathrm{Re}(z) - b)p(bp+q) + (bp+q)^2\right) = V_{\mathrm{BH}}^{(p,q)}\,. \tag{4.63}$$

(iii) Scale transformations act as

$$z \to a^2 z\,, \begin{pmatrix} p \\ q \end{pmatrix} \to \begin{pmatrix} \frac{1}{a}p \\ aq \end{pmatrix}\,. \tag{4.64}$$

We then have

$$V_{\mathrm{BH}}^{(p,q)} \to \frac{1}{a^2}\frac{1}{2\,\mathrm{Im}(z)}\left(a^4|z|^2\frac{1}{a^2}p^2 + 2a^2\,\mathrm{Re}(z)pq + a^2q^2\right) = V_{\mathrm{BH}}^{(p,q)}\,. \tag{4.65}$$

Since the first term, and hence also the second term, of the black hole potential is invariant under the generators of $\mathrm{SL}(2,\mathbb{R})$, it is invariant under transformations of the full group, as was to be shown.

The Einstein equations can be written as a function of the black hole potential, namely

$$R_{tt} = e^{6U}\tau^4 V_{\mathrm{BH}}\,, \tag{4.66}$$

$$R_{\tau\tau} = \ddot{U} - 2\dot{U}^2 = -e^{2U}V_{\mathrm{BH}} + |\dot{z}|^2/2(\mathrm{Im}(z))^2\,, \tag{4.67}$$

$$R_{\theta\theta} = \tau^2\ddot{U} = \tau^2 e^{2U}V_{\mathrm{BH}}\,, \tag{4.68}$$

where dots denote derivatives with respect to τ. The equation for $R_{\varphi\varphi}$ differs from the $R_{\theta\theta}$ equation by a factor $\sin^2\theta$. Hence we find the two independent conditions

$$\ddot{U} = e^{2U}V_{\mathrm{BH}}\,, \tag{4.69}$$

$$\dot{U}^2 = e^{2U}V_{\mathrm{BH}} - \frac{|\dot{z}|^2}{(2\,\mathrm{Im}(z))^2}\,. \tag{4.70}$$

4.2.3 Scalar equations of motion

One can also derive the scalar equations of motion by varying the action with respect to z. It turns out that we can also obtain these equations from varying an action functional resembling a classical mechanics systems with variables $U(\tau), z(\tau)$:

$$S[U,z] = \int \mathrm{d}\tau \left(\dot{U}^2 + \frac{1}{4(\mathrm{Im}(z))^2}|\dot{z}|^2 + e^{2U}V_{\mathrm{BH}} \right) . \tag{4.71}$$

The Noether theorem then gives a conserved quantity, an "energy"

$$\mathcal{E} \equiv \dot{U}^2 + \frac{|\dot{z}|^2}{(2\,\mathrm{Im}(z))^2} - e^{2U}V_{\mathrm{BH}} \,, \tag{4.72}$$

which is independent of τ. The above equation for $\dot{U}^2$ then implies we only consider solutions with zero "energy" in this text.

5 Supersymmetry of black holes

In this section, we focus on supersymmetric solutions of supergravity theories and apply these ideas to our black hole attractor solutions. We start off by discussing the general concepts of Killing spinors and BPS solutions. Then, we return to our dilaton black hole and show that this solution has residual supersymmetry. Afterwards, we discuss the concept of central charge, since another formulation of BPS solution, for massive representations, states that the mass is equal to the absolute value of the central charge. Central charges are related to the black hole potential and allow us to elegantly show that the general black hole solutions with charges (p,q) and (p',q') are also BPS solutions.

5.1 Killing spinors and BPS solutions

The above black hole solutions are formulated in $\mathcal{N}=2$ supergravity, a theory with two independent supersymmetries. Generally, this does *not* imply that solutions of the theory are invariant under supersymmetry transformations as well. If this is the case, however, then the solution is said to be a *Bogomol'nyi-Prasad-Sommerfeld* or *BPS solution* (or simply 'supersymmetric solution'). BPS solutions are solutions which are invariant under a subalgebra of the supersymmetric algebra which contains at least one fermionic generator. In essence, BPS solutions are solutions of the theory which themselves carry a residual amount of *global* supersymmetry.

There exists a prescription to determine such special solutions. The essential tools to work with are the *Killing spinors*. These are a finite subset of the spinor functions for which the supersymmetry transformations leave the solution invariant. The Killing spinors contain a set of constant parameters and hence determine the residual (global) supersymmetry of the solution. One can derive Killing spinor conditions (of which the solutions are the desired Killing spinors) from the fermion transformation rules. With a classical solution at hand, one can write schematically the local supersymmetry transformations as

$$\delta_\epsilon B(x) = \bar{\epsilon}(x) f(B)\ F(x) + \dots\,, \quad \delta_\epsilon F(x) = g(B)\epsilon(x) + \dots\,, \tag{5.1}$$

where $B(x)$ and $F(x)$ represent boson and fermion fields, respectively. The dots denote higher-order terms involving fermion fields only. If the solution has residual supersymmetry, this means that the set of equations $\delta_\epsilon F(x) = 0$, $\delta_\epsilon B(x) = 0$ has a non-trivial solution for $\epsilon(x)$. However, fermion fields vanish in a classical solution, and hence the equation $\delta_\epsilon B(x)$ is certainly satisfied. Since the higher-order terms vanish as well, only the linear term in the $\delta_\epsilon F(x)$ equation remains. Hence a BPS solution is a classical solution where the linearized fermion supersymmetry transformations vanish:

$$\delta_\epsilon F(x)|_{\text{lin}} \equiv g(B)\epsilon(x) = 0\,. \tag{5.2}$$

If there are n_Q linearly independent solutions to this set of equations, then we say the solution preserves n_Q supercharges. The solution is then said to be $\frac{n_Q}{\mathcal{N}}$-BPS. An important point to keep in mind is that the original symmetries are local, while the preserved ones are global. The goal in the remainder of this section is to show that the dilaton black hole is a $\frac{1}{2}$-BPS solution by considering the linearized fermion supersymmetry transformations. However, we will need some elements from $\mathcal{N} = 2$ supergravity for this.

5.1.1 Basics of $\mathcal{N} = 2$ supergravity

Before we dive into the details of showing that the dilaton black hole is a BPS solution, we provide a very brief overview of the basics of $\mathcal{N} = 2, D = 4$ supergravity. Note that the details of $\mathcal{N} = 2$ supergravity are not essential to the story here, so we will necessarily be too brief in this section. In particular, we do not discuss gauging, moment maps, the relevance of hypermultiplets,... since these ingredients are absent in our black hole attractor solutions.

In matter-coupled $\mathcal{N} = 2$ supergravity, gravity is coupled to n_V vector (gauge) multiplets and n_H hypermultiplets. Note that in our model above, we have $n_V = 1$, $n_H = 0$. This implies that the bosonic sector has, apart from the graviton field, also $n_V + 1$ gauge fields A^I_μ, $I = 0, \dots, n_V$ and $n_V + 1$ scalars X^I. These scalars can be parametrized by homogeneous coordinates, such that n_V complex scalars z^α remain. The scalar manifold is a direct product of two target spaces determined by the scalars in the gauge multiplets and hypermultiplets, respectively. The n_V complex scalars define a *special Kähler manifold*. This is a Kähler manifold with *symplectic structure* $\mathrm{Sp}(2(n_V + 1))$ which originates from the $\mathcal{N} = 2$ supersymmetry connecting the scalars with the gauge fields. The scalars of the hypermultiplets, on the other hand, define a quaternionic-Kähler manifold. The kinetic terms of gauge vectors only depend on the special Kähler manifold via a matrix $\mathcal{N}_{IJ}(z, \bar{z})$ given in equation (21.5) of the book. From the Kähler potential $\mathcal{K}$, one can find the metric on the scalar manifold, the Kähler metric, by[3]

$$\mathcal{K}_{\alpha\bar{\beta}} \equiv \partial_\alpha \partial_{\bar{\beta}} \mathcal{K}\,. \tag{5.3}$$

[3]We use α, $\bar{\beta}$ to refer to the complex scalars, with regular Greek letters denoting the scalars, and barred Greek letters denoting their complex conjugates. The notation in (5.3) defines subscripts of $\mathcal{K}$ to denote derivatives of $\mathcal{K}$ with respect to these scalars.

We do not discuss how, in general, the Kähler potential can be found. The supergravity theory can be expressed using this symplectic structure, in its so-called *symplectic formulation*, but we do not provide further details other than those needed for the exercises below. The action and transformation rules can be found in Section 21.3 of the book. Of particular importance for our discussion of BPS solutions are the bosonic parts of the fermion supersymmetry transformation rules, see equation (21.42).

5.1.2 The dilaton black hole as $\frac{1}{2}$-BPS solution

We now show that the dilaton black hole is a BPS solution and preserves one supersymmetry. For our model, we can ignore hypermultiplets and gaugings, such that the linearized fermion supersymmetry transformations become

$$\delta\psi_\mu^A = \left(\partial_\mu + \tfrac{1}{4}\omega_\mu{}^{ab}\gamma_{ab} - \tfrac{1}{2}i\mathcal{A}_\mu\right)\epsilon^A - \tfrac{1}{16}\gamma^{ab}T^-_{ab}\varepsilon^{AB}\gamma_\mu\epsilon_B \tag{5.4}$$

$$\delta\chi^\alpha_A = \gamma^\mu\partial_\mu z^\alpha\epsilon_A - \tfrac{1}{2}\kappa^{-2}\mathcal{K}^{\alpha\overline{\beta}}G^-_{ab\,\overline{\beta}}\gamma^{ab}\varepsilon_{AB}\epsilon^B\,, \tag{5.5}$$

where the frame fields and connections are given in equations (22.70) and (22.71) of the book. We used indices A, B to refer to the R-symmetry, and also introduced

$$\mathcal{A}_\mu = \tfrac{1}{2}i\kappa^2\left(\partial_\mu z^\alpha\partial_\alpha\mathcal{K} - \partial_\mu\overline{z}^{\overline{\alpha}}\partial_{\overline{\alpha}}\mathcal{K}\right) \tag{5.6}$$

$$T^-_{ab} = -4X^I\,\mathrm{Im}(\mathcal{N}_{IJ})F^{-J}_{ab} \tag{5.7}$$

$$G^-_{ab\,\overline{\beta}} = \overline{\nabla}_{\overline{\beta}}\overline{X}^I\,\mathrm{Im}(\mathcal{N}_{IJ})F^{-J}_{ab}\,, \tag{5.8}$$

where 2-forms like F^{-J}_{ab} are defined by equation (2.12). We parametrized $X^I = yZ^I(z)$, with y related to the Kähler potential by $y = e^{\kappa^2\mathcal{K}/2}$. The Kähler covariant derivative is defined to act as

$$\nabla_\alpha Z^I = \left[\partial_\alpha + \kappa^2(\partial_\alpha\mathcal{K})\right]Z^I\,. \tag{5.9}$$

As before, we do not discuss the general ideas behind all of the above and limit ourselves to our own model with $n_V = 1$ and one coordinate z. For this, we can use the results of Exercise 20.18 and Exercise 20.19 from the book, which tells us that

$$Z^0 = 1\,,\quad Z^1 = i\,,\quad e^{-\kappa^2\mathcal{K}} = 4\,\mathrm{Im}(z)\,,\quad \mathcal{K}_{z\overline{z}} = (2\kappa\,\mathrm{Im}(z))^2\,,\quad \mathcal{N}_{IJ} = -\kappa^2 z\delta_{IJ}\,. \tag{5.10}$$

Note that indices are now restricted to $I = 0, 1$, and $\alpha = z$ has only one value. The scalar y has the value $y = 1/(2\sqrt{\mathrm{Im}(z)})$. We now look for the possibility of the dilaton black hole being a solution of vanishing fermion transformations. This means that z is imaginary, $F^0_{\mu\nu}$ is electric, and $F^1_{\mu\nu}$ is magnetic, so

$$z = ie^{-2\phi}\,,\quad F^0_{i0} = \partial_i\psi\,,\quad \tilde{F}^1_{i0} = ie^{2\phi}\partial_i\chi\,. \tag{5.11}$$

The ansatz for $F^1_{\mu\nu}$ ensures that the dual, $G_{\mu\nu 1}$, is purely electric. Plugging everything into the Killing conditions $\delta\psi^A_\mu = 0$, $\delta\chi^\alpha_A = 0$ yields[4]

$$0 = \tfrac{1}{2}e^{2U}\gamma_{\hat{0}}\gamma_{\hat{i}}(\partial_i U)\epsilon^A - \tfrac{1}{4}\gamma^{\hat{i}}e^{U-\phi}\left(\partial_i\psi + e^{2\phi}\partial_i\chi\right)\varepsilon^{AB}\epsilon_B\,, \tag{5.12}$$

$$0 = \partial_i\epsilon^A - \tfrac{1}{2}\gamma_{\hat{i}\hat{j}}(\partial_j U)\epsilon^A + \tfrac{1}{4}\gamma^{\hat{0}\hat{j}}e^{-U-\phi}\left(\partial_j\psi + e^{2\phi}\partial_j\chi\right)\gamma_{\hat{i}}\varepsilon^{AB}\epsilon_B\,, \tag{5.13}$$

$$0 = i\gamma^{\hat{i}}e^U\partial_i e^{-2\phi}\epsilon_A + ie^{-3\phi}\left(\partial_i\psi - e^{2\phi}\partial_i\chi\right)\gamma^{\hat{0}\hat{i}}\varepsilon_{AB}\epsilon^B\,. \tag{5.14}$$

Note that we are assuming the Killing spinors to be static, since our metric is static as well. The first and second equation are $\delta\psi^A_0$ and $\delta\psi^A_i$, respectively. We combine the first equation and the C-conjugate of the last to

$$\gamma^{\hat{i}}\epsilon^A\partial_i e^{U+\phi} = \gamma^{\hat{i}}(\partial_i\psi)\gamma_{\hat{0}}\varepsilon^{AB}\epsilon_B\,, \quad \gamma^{\hat{i}}\epsilon^A\partial_i e^{U-\phi} = \gamma^{\hat{i}}(\partial_i\chi)\gamma_{\hat{0}}\varepsilon^{AB}\epsilon_B\,. \tag{5.15}$$

The remaining Killing condition becomes

$$(\partial_i - \tfrac{1}{2}\partial_i U)\epsilon^A = 0\,, \tag{5.16}$$

since we only consider solutions depending on r. We solve the Killing conditions with

$$\psi = \pm e^{U+\phi}\,, \quad \chi = \pm e^{U-\phi}\,, \quad \epsilon^A = \pm\gamma_{\hat{0}}\varepsilon^{AB}\epsilon_B\,, \quad \epsilon^A = e^{U/2}\epsilon^A_{(0)}\,. \tag{5.17}$$

Here, $\epsilon^A_{(0)}$ is a constant spinor, and the signs are correlated. Hence we have two distinct sets of solutions. We can now readily make the link with our earlier discussion of the dilaton black hole, by setting $\psi = \pm H_1^{-1}$ and $\chi = \pm H_2^{-1}$. The solution with the upper (positive) sign corresponds to negative electric and positive magnetic charge, while the solution with the lower (negative sign) has a positive electric and negative magnetic charge: see Exercise 22.20. Hence we have shown that for the dilaton black hole as classical solution, there exists a non-trivial configuration of the supersymmetry parameters for which the linearized fermion supersymmetry transformations vanish. In essence, we have shown that the dilaton black hole is a BPS solution.

We can gain even more insight by changing the basis of the spinors from $\{\epsilon_1, \epsilon_2\}$ to $\{\epsilon_+, \epsilon_-\}$, where

$$\epsilon_\pm = \epsilon_2 \pm \gamma^{\hat{0}}\epsilon^1\,, \tag{5.18}$$

which are still chiral. This basis is convenient, since it directly shows that one of these two Killing spinors must vanish. Let us show this for the solution with the upper signs in equation (5.17). Then we find

$$\epsilon_+ = \epsilon_2 + \gamma^{\hat{0}}\epsilon^1 = \epsilon_2 + \gamma^{\hat{0}}\gamma_{\hat{0}}\varepsilon^{12}\epsilon_2 = 0\,, \tag{5.19}$$

where we used that $\varepsilon^{12} = -1$. A similar analysis shows that in the other solution, we have $\epsilon_- = 0$. Hence the solution with the two positive signs preserves the ϵ_- supersymmetry, while the solution with two negative signs preserves the ϵ_+ supersymmetry.

In Exercise 22.20, we said that there are four different configurations for the signs in the field strenghts for the dilaton black hole solution. Here, only the two possibilities with opposite sign for the charges appear. However, in Section 7, we will explain that the other two sign configurations do arise as $\frac{1}{4}$-BPS solutions in $\mathcal{N} = 4$ supergravity.

[4] We follow the book's convention where hatted indices denote frame indices.

5.2 Central charge

We now dedicate a few words on the concept of central charge, its relation with BPS solutions and how it is relevant to our work. A *central charge* is a new operator $\mathcal{Z}$ which, in extended supersymmetry, can be inserted in the anti-commutator of two supercharges of identical chirality. For $\mathcal{N}=2$, the anti-commutation relations become:

$$\{Q_{\alpha i}, Q_{\beta j}\} = -\tfrac{1}{2}\varepsilon_{ij} P_{L\alpha\beta}\mathcal{Z}\,, \quad \{Q_\alpha{}^i, Q_\beta{}^j\} = -\tfrac{1}{2}\varepsilon^{ij} P_{R\alpha\beta}\overline{\mathcal{Z}}\,. \tag{5.20}$$

This operator is 'central' in the mathematical sense: it commutes with all other operators. Adding central charges does not modify other anti-commutation relations between supercharges.

One can derive, from the algebra, that $P^0 \geq |\mathcal{Z}|$, implying that $M \geq |\mathcal{Z}|$ in case of massive representations. This is called the *positivity bound*, or also the *BPS bound*. Solutions that satisfy this bound are also called BPS solutions and can be shown to be supersymmetric as well. Central charges are necessary to allow for massive supersymmetric solutions. We will see that the BPS bound is satisfied by the dilaton black hole, as expected.

For solutions with electric and magnetic charges, we can define central charges via equation (21.52) in the book, where it is shown that

$$\mathcal{Z} = 2\kappa^{-2}(X^I q_I - F_I p^I)\,, \tag{5.21}$$

where F_I can be obtained from X^I using $F_I = \mathcal{N}_{IJ}X^J$. For a model with $n_V = 1$, we can again use the information provided in Exercise 20.18 and Exercise 20.19: see equation (5.10). Therefore, we have

$$X^I = \frac{1}{2}\begin{pmatrix} 1/\sqrt{\mathrm{Im}(z)} \\ i/\sqrt{\mathrm{Im}(z)} \end{pmatrix}\,, \quad F_I = \mathcal{N}_{IJ}X^J = -\frac{z}{2}\begin{pmatrix} 1/\sqrt{\mathrm{Im}(z)} \\ i/\sqrt{\mathrm{Im}(z)} \end{pmatrix}\,. \tag{5.22}$$

Substituting in equation (5.21), with $q_0, p^0 \to q, p$ and $q_1, p^1 \to q', p'$, we find

$$\mathcal{Z} = \frac{\kappa^{-2}}{\sqrt{\mathrm{Im}(z)}}\left[(q + iq') + z(p + ip')\right]\,. \tag{5.23}$$

Let us now consider the dilaton black hole and put $q' = 0 = p$, and $z = ie^{-2\phi}$. Then the central charge reads

$$\mathcal{Z} = \kappa^{-2}(qe^{\phi} - p'e^{-\phi})\,. \tag{5.24}$$

We can now use the result from Exercise 22.20, which tells us that $q = \pm|q|$ and $p' = \mp|p'|$, to find

$$\mathcal{Z} \overset{!}{=} \pm\kappa^{-2}\left(|q|e^{\phi} + |p'|e^{-\phi}\right)\,. \tag{5.25}$$

Note that the previous equation is only valid if the signs are correlated, hence the exclamation mark. This means that our discussion holds for dilaton black holes where electric and magnetic charge have opposite signs. The central charge at infinity $\mathcal{Z}_\infty$ is defined by taking the limit of $\mathcal{Z}$ for $r \to +\infty$ and amounts to replacing ϕ with ϕ_0. Comparing with equation (3.24), we see that the solution with positive signs satisfies $M = \mathcal{Z}_\infty$, while the solution with negative signs satisfies $M = -\mathcal{Z}_\infty$. In essence, the dilaton black hole, with charges having opposite signs, satisfies

$M = |\mathcal{Z}_\infty|$. This equality is another proof that the dilaton black hole is a BPS solution. Note that this equality can also be obtained by studying the transformation of the frame field component $e_0^{\hat{i}}$ and putting this equal to zero, since we are looking at supersymmetric solutions. We do not provide the details here.

5.2.1 Black hole potential

The importance of central charge for black hole attractor solutions goes further than the proof that $M = |\mathcal{Z}_\infty|$. It turns out that the black hole potential is related to the central charge, which we now show for the multi-component $\mathcal{N} = 2$ model.

Exercise 22.27

Recall the general definition of the central charge as a function of the electric and magnetic charge:

$$\mathcal{Z} = 2\kappa^{-2}\left(X^I q_I - F_I p^I\right) . \tag{5.26}$$

As already mentioned before, we have $F_I = \mathcal{N}_{IJ}X^J$, and hence we can write

$$\mathcal{Z} = 2\kappa^{-2}X^I\left(q_I - \mathcal{N}_{IJ}p^J\right) . \tag{5.27}$$

Hence we also have

$$\overline{\mathcal{Z}} = 2\kappa^{-2}\overline{X}^I\left(q_I - \overline{\mathcal{N}}_{IJ}p^J\right) . \tag{5.28}$$

We will also need $\nabla_\alpha\mathcal{Z}$, with ∇_α denoting the Kähler covariant derivative, for which we need a bit more information on the symplectic formulation. In symplectic notation, the values of $\mathcal{Z}$ and $\nabla_\alpha\mathcal{Z}$ are given by

$$\mathcal{Z} = 2\langle V, \Gamma\rangle, \quad \nabla_\alpha\mathcal{Z} = 2\langle\nabla_\alpha V, \Gamma\rangle \quad V = \begin{pmatrix} X^I \\ F_I \end{pmatrix} , \quad \Gamma = \begin{pmatrix} p^I \\ q_I \end{pmatrix} , \tag{5.29}$$

where V, Γ are *symplectic vectors* and $\langle\cdot,\cdot\rangle$ denotes the *symplectic product* defined to act as

$$\langle V, \overline{V}\rangle = X^I\overline{F}_I - F_I\overline{X}^I . \tag{5.30}$$

This indeed gives the value for $\mathcal{Z}$, if we absorb a factor κ^2 appropriately. The final piece of the puzzle is the information that

$$\nabla_\alpha V = \begin{pmatrix} \nabla_\alpha X^I \\ \nabla_\alpha F_I \end{pmatrix} , \quad \nabla_\alpha F_I = \overline{\mathcal{N}}_{IJ}\nabla_\alpha X^J . \tag{5.31}$$

We can then find the value of $\nabla_\alpha\mathcal{Z}$:

$$\nabla_\alpha\mathcal{Z} = 2\kappa^{-2}\left(\nabla_\alpha X^I q_I - \nabla_\alpha F_I p^I\right) \tag{5.32}$$

$$= 2\kappa^{-2}\nabla_\alpha X^I\left(q_I - \overline{\mathcal{N}}_{IJ}p^I\right) , \tag{5.33}$$

from which it follows that

$$\overline{\nabla}_{\overline{\alpha}}\overline{\mathcal{Z}} = 2\kappa^{-2}\overline{\nabla}_{\overline{\alpha}}\overline{X}^I\left(q_I - \mathcal{N}_{IJ}p^I\right) . \tag{5.34}$$

We now make a calculation which will introduce the black hole potential. Observe that

$$\begin{aligned}\kappa^2\mathcal{Z}\overline{\mathcal{Z}} + \mathcal{K}^{\alpha\overline{\beta}}\nabla_\alpha\mathcal{Z}\overline{\nabla}_{\overline{\beta}}\overline{\mathcal{Z}} &= 4\kappa^{-4}\Big[\kappa^2 X^I\overline{X}^K\left(q_I - \mathcal{N}_{IJ}p^J\right)\left(q_K - \overline{\mathcal{N}}_{KL}p^L\right)\\ &\qquad + \mathcal{K}^{\alpha\overline{\beta}}\nabla_\alpha X^I\overline{\nabla}_{\overline{\beta}}\overline{X}^K\left(q_I - \overline{\mathcal{N}}_{IJ}p^J\right)\left(q_K - \mathcal{N}_{KL}p^L\right)\Big] .\\ &= 4\kappa^{-4}\ \Big[\kappa^2 X^I\overline{X}^K\left(q_I - \mathcal{N}_{IJ}p^J\right)\left(q_K - \overline{\mathcal{N}}_{KL}p^L\right)\\ &\qquad + \mathcal{K}^{\alpha\overline{\beta}}\nabla_\alpha X^K\overline{\nabla}_{\overline{\beta}}\overline{X}^I\left(q_K - \overline{\mathcal{N}}_{KL}p^L\right)\left(q_I - \mathcal{N}_{IJ}p^J\right)\Big] .\\ &= 4\kappa^{-4}\left(q_I - \mathcal{N}_{IJ}p^J\right)\left(q_K - \overline{\mathcal{N}}_{KL}p^L\right)\Big[\kappa^2\overline{X}^K X^I + \mathcal{K}^{\alpha\overline{\beta}}\nabla_\alpha X^K\overline{\nabla}_{\overline{\beta}}\overline{X}^I\Big] ,\end{aligned} \tag{5.35}$$

where in the first to second equality, we simply relabeled dummy indices $I \leftrightarrow K$ and $J \leftrightarrow L$ in the second term in order to get a common factor out. For the second factor in square brackets, we can use the result of Exercise 20.29, which tells us that

$$-\tfrac{1}{2}\left(I^{-1}\right)^{IJ} = \left(\kappa^2\overline{X}^I X^J + \mathcal{K}^{\alpha\overline{\beta}}\nabla_\alpha X^I\overline{\nabla}_{\overline{\beta}}\overline{X}^J\right) , \quad I \equiv \mathrm{Im}(\mathcal{N}) . \tag{5.36}$$

Then equation (5.35) becomes, after taking care of the redefinitions with κ^2:

$$\kappa^2\mathcal{Z}\overline{\mathcal{Z}} + \mathcal{K}^{a\overline{\beta}}\nabla_\alpha\mathcal{Z}\overline{\nabla}_{\overline{\beta}}\overline{\mathcal{Z}} = -4\kappa^{-2}\left(q_I - \mathcal{N}_{IJ}p^J\right)\left(q_K - \overline{\mathcal{N}}_{KL}p^L\right)\left(I^{-1}\right)^{KI} . \tag{5.37}$$

By explicitly expanding the matrix product, one can show that the right hand side is equal to

$$\kappa^2\mathcal{Z}\overline{\mathcal{Z}} + \mathcal{K}^{a\overline{\beta}}\nabla_\alpha\mathcal{Z}\overline{\nabla}_{\overline{\beta}}\overline{\mathcal{Z}} = 2\kappa^{-2}\begin{pmatrix}p & q\end{pmatrix}\mathcal{M}\begin{pmatrix}p\\ q\end{pmatrix} , \tag{5.38}$$

where $\mathcal{M}$ is the matrix defined in equation (4.25). Recalling the definition of the black hole potential in equation (4.44), we then find

$$(4\pi)^2 V_{\mathrm{BH}} = \tfrac{1}{4}\kappa^4\mathcal{Z}\overline{\mathcal{Z}} + \tfrac{1}{4}\kappa^2\mathcal{K}^{\alpha\overline{\beta}}\nabla_\alpha\mathcal{Z}\overline{\nabla}_{\overline{\beta}}\overline{\mathcal{Z}} . \tag{5.39}$$

We can further simplify this. Note that $\nabla_\alpha\overline{\mathcal{Z}} = 0$ and $\nabla_\alpha(\mathcal{Z}\overline{\mathcal{Z}}) = \partial_\alpha(\mathcal{Z}\overline{\mathcal{Z}})$, such that

$$\nabla_\alpha\mathcal{Z} = 2\sqrt{\frac{\mathcal{Z}}{\overline{\mathcal{Z}}}}\partial_\alpha\mathcal{Z} . \tag{5.40}$$

By taking the conjugate of this result, substituting in the above, and using $\kappa^2 = 8\pi G$, we end up with

$$G^{-2}V_{\mathrm{BH}} = \mathcal{Z}\overline{\mathcal{Z}} + 4\kappa^{-2}\mathcal{K}^{\alpha\overline{\beta}}\partial_\alpha|\mathcal{Z}|\partial_{\overline{\beta}}|\mathcal{Z}| . \tag{5.41}$$

6 First order gradient flow equations

We now return to the general black hole solutions discussed before. Using the black hole potential, we will derive a set of first-order differential equations governing the dynamics of the black hole solution, which eventually will allow us to see the attractor phenomenon emerging in the more general solution with charges (p, q) and (p', q'). The gradient flow equations then allow us to show that $M = |\mathcal{Z}_\infty|$, such that the general black hole attractors are supersymmetric as well.

To start, first recall that Exercise 20.18 tells us that

$$\mathcal{K}_{z\bar{z}} = (2\kappa \,\mathrm{Im}(z))^{-2} \,. \tag{6.1}$$

Substituting in the general result for the black hole potential, equation (5.41), then gives

$$G^{-2} V_{\mathrm{BH}} = |\mathcal{Z}|^2 + 16(\mathrm{Im}(z))^2 \partial_z |\mathcal{Z}| \partial_{\bar{z}} |\mathcal{Z}| \,. \tag{6.2}$$

The scalar equation of motion could be found from an action resembling a classical mechanics system: see equation (4.71), which depended on V_{BH}. Substituting the above result for V_{BH} now yields

$$S[U, z] = \int \mathrm{d}\tau \left(\dot{U}^2 + \frac{|\dot{z}|^2}{(2\,\mathrm{Im}(z))^2} + G^2 e^{2U} |\mathcal{Z}|^2 + 16 G^2 e^{2U} (\mathrm{Im}(z))^2 \partial_z |\mathcal{Z}| \partial_{\bar{z}} |\mathcal{Z}| \right) . \tag{6.3}$$

We can combine the first and third, and second and last term and complete the squares. The additional terms appearing can be recast into total derivatives and integrated out. The result is then

$$S[U, z] = \int \mathrm{d}\tau \left[\left(\dot{U} + G e^U |\mathcal{Z}| \right)^2 + \frac{1}{4(\mathrm{Im}(z))^2} \left| \dot{z} + 8G (\mathrm{Im}(z))^2 e^U \partial_{\bar{z}} |\mathcal{Z}| \right|^2 \right] - 2 G e^U \left| \mathcal{Z} \right| \Big|_{\tau=0}^{\tau=+\infty} . \tag{6.4}$$

This is now an extremely easy functional to extremize: since the Lagrangian is a sum of squares, the condition to be at a minimum yields the equations

$$\begin{cases} \dot{z} &= -8 G e^U (\mathrm{Im}(z))^2 \partial_{\bar{z}} |\mathcal{Z}| \\ \dot{U} &= -G e^U |\mathcal{Z}| \,. \end{cases} \tag{6.5}$$

A solution of these equations automatically solves the scalar equation of motion and the Einstein equations, which are equations on $\dot{U}$ and $\ddot{U}$. Again, the solutions we are interested in have zero "energy" $\mathcal{E}$, as defined before. The above set of first-order coupled differential equations exhibit the structure of a *gradient flow*. Hence the dynamics of these black holes can be reduced to a system of gradient "flow" equations for the variables z and U with "time" τ.

We now prove the equality $M = |\mathcal{Z}|_\infty$ from the gradient flow equations. Since the solution becomes flat at large distances (which correspond to $\tau = 0$), we employ the normalization $e^{2U(0)} = 1$ such that $g_{tt} = -1$ at spatial infinity. The above differential equations tell us that g_{tt} flows monotonically upwards and approaches zero as we approach the horizon, as expected. At large distance, we can expand this metric component in powers of $1/r$. From our recap on black hole solutions, we know that the first-order correction contains the mass M, such that

$$-g_{tt} = e^{2U} \approx 1 - 2MG\tau = 1 + \frac{\mathrm{d}\left(e^{2U}\right)}{\mathrm{d}\tau}\Big|_{\tau=0} \tau \,. \tag{6.6}$$

Making use of the equation for $\dot{U}$, we obtain that

$$M = -\frac{1}{G} e^{2U(0)} \dot{U}(0) = |\mathcal{Z}_\infty| \,. \tag{6.7}$$

This is precisely the BPS condition. Hence we can conclude that the extremal charged black hole solutions are BPS solutions.

The fixed point of the system of gradient flow equations is located at an extremum of $|\mathcal{Z}|$, *i.e.* where $\partial_\alpha |\mathcal{Z}| = 0$. Since $\dot{U}$ has to be zero as well, this is only possible if $e^{2U} \to 0$, which is at the horizon $\tau \to \infty$. Hence the scalar field flows from a value determined by a boundary condition at spatial infinity, and approaches a constant value at the black hole's horizon which is an extremum of $|\mathcal{Z}|$. Therefore, the fixed point must be a minimum of $|\mathcal{Z}|$. Indeed, one can check explicitly that this is true for the dilaton black hole by looking at equation (5.25). So the scalar dynamics has precisely the black hole attractor mechanism which we first encountered for the dilaton black hole.

The area of the horizon is determined by the minimum value of $|\mathcal{Z}|$. This also corresponds to the minimal value of the black hole potential, since at $\partial_z |\mathcal{Z}| = 0$, V_{BH} has a value $V_{\mathrm{BH}} = G^2 |\mathcal{Z}|^2_{\mathrm{min}}$. To check this, note that in Exercise 22.21 we found that the area of a 2-sphere at a fixed distance r is $A(\tau) = 4\pi e^{-2U(\tau)}/\tau^2$, with $\tau = r^{-1}$. We can rewrite this in terms of the central charge by invoking the flow equations. First note that

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \left(e^{-U} \right) = -\dot{U} e^{-U} = G|\mathcal{Z}| \xrightarrow{\tau \to \infty} G|\mathcal{Z}|_{\mathrm{min}} \,. \tag{6.8}$$

Now, the limit of $A(\tau)$ for $\tau \to \infty$ (*i.e.* the horizon) is undefined, since both numerator and denominator tend to zero. We can fix this by using l'Hôpitals' rule twice, and find

$$\lim_{\tau \to \infty} A(\tau) = 4\pi \lim_{\tau \to \infty} \left(\frac{-2\dot{U} e^{-2U}}{2\tau} \right) = 4\pi G |\mathcal{Z}| \lim_{\tau \to \infty} \left(\frac{\mathrm{d}}{\mathrm{d}\tau} e^{-U} \right) = 4\pi G^2 |\mathcal{Z}|^2_{\mathrm{min}} \tag{6.9}$$

The area of the black hole horizon is therefore

$$A = 4\pi G^2 |\mathcal{Z}|^2_{\mathrm{min}} = 4\pi V_{\mathrm{BH,min}} \,. \tag{6.10}$$

We now verify the result of Exercise 22.21 with this formula. The central charge for the dilaton black hole was given in equation (5.25). This is a real quantity, and the minimum of $|\mathcal{Z}|$ lies at the value of the dilaton at the horizon: $e^{-2\phi} = e^{-2\phi_h} = |q/p'|$: see the derivation around equation (3.26). We then find

$$|\mathcal{Z}|_{\mathrm{min}} = \kappa^{-2} e^{\phi_h} |q| \,. \tag{6.11}$$

Taking the square, and using the value of $e^{-2\phi_h}$, we indeed find that $A = |qp'|/(4\pi)$, as we also showed by a direct calculation without the central charge in Section 3.3.

7 Outlook and conclusion

In this final section, we give a brief overview of possible extensions for future work as well as a few interesting topics which are related to supersymmetric black holes.

7.1 Generalization to multiple vector multiplets

We can easily extend our discussion of the attractor mechanism to the general situation of $\mathcal{N} = 2$ supergravity theories coupled to n_V abelian vector multiplets. This means we have $n_V + 1$ gauge fields and field strenghts $F^I_{\mu\nu}$, and n_V complex scalars z^α. The action, after rescaling the Kähler metric with κ^2, reads

$$S = \frac{1}{2\kappa^2}\int \mathrm{d}^4x \left[\sqrt{-g}\left(R - 2g^{\mu\nu}\mathcal{K}_{\alpha\bar\beta}\partial_\mu z^\alpha \partial_\nu \overline{z}^{\overline{\beta}} + \tfrac{1}{2}\operatorname{Im}(\mathcal{N}_{IJ})F^I_{\mu\nu}F^{\mu\nu J}\right) - \tfrac{1}{4}\operatorname{Re}(\mathcal{N}_{IJ})\varepsilon^{\mu\nu\rho\sigma}F^I_{\mu\nu}F^J_{\rho\sigma}\right] . \tag{7.1}$$

The index I has values $I = 0, \ldots, n_V$, and $\alpha, \bar\beta$ label the complex scalars and their complex conjugates, respectively. Some of the results in previous sections were already derived in this general situation of n_V vector multiplets. It turns out that minimal changes are needed to reproduce the analysis of the attractor phenomenon in this more general case. A first modification is for the stress tensor:

$$\kappa^2 T_{\mu\nu} = -\operatorname{Im}(\mathcal{N}_{IJ})\left(F^I_{\mu\rho}F_\nu{}^{\rho J} - \frac{1}{4}g_{\mu\nu}F^I_{\rho\sigma}F^{J\rho\sigma}\right) + \mathcal{K}_{\alpha\overline{\beta}}\left(\partial_\mu z^\alpha \partial_\nu \overline{z}^{\overline{\beta}} + \partial_\nu z^\alpha \partial_\mu \overline{z}^{\overline{\beta}} - g_{\mu\nu}\partial_\rho z^\alpha \partial^\rho \overline{z}^{\overline{\beta}}\right) . \tag{7.2}$$

The source terms of the Ricci tensor are derived similar to the $n_V = 1$ case, but we modify

$$\frac{|\dot{z}|^2}{2(\operatorname{Im}(z))^2} \to 2\mathcal{K}_{\alpha\bar\beta}\dot{z}^\alpha \dot{\overline{z}}^{\overline{\beta}} . \tag{7.3}$$

Another modification is in the 'classical action', which now reads

$$S\left[U, \{z\}, \{\overline{z}\}\right] = \int \mathrm{d}\tau \left(\dot{U}^2 + \mathcal{K}_{\alpha\overline{\beta}}\dot{z}^\alpha \dot{\overline{z}}^{\overline{\beta}} + e^{2U}V_{\mathrm{BH}}\right) . \tag{7.4}$$

In Exercise 22.27, we already derived V_{BH} in the more general case of n_V multiplets. Varying the 'classical' action again gives a set of first-order differential flow equations. By copying the analysis of the $n_V = 1$ model with these changes, we can easily derive the attractor phenomenon of supersymmetric black holes in $\mathcal{N} = 2$ supergravity with n_V abelian vector multiplets.

7.2 Related theories and solutions

Let us first give a few possible directions one could take to further investigate supersymmetric black holes and black hole attractors. There are a few options which are clear by reconsidering the assumptions we made in this work. For example, no extremal black holes were considered for most of the work considered here. It turns out that non-extremal black holes always break supersymmetry [5]. Other possibilities are to include multiple dilaton black holes at different locations, generalizing the Papapetrou-Majumdar solution to include dilatons, include a gauge group in the theory, or considering coupling to hypermultiplets. There also exist non-BPS but still extremal black hole solutions, for which the role of the central charge in the flow equations is replaced by a so-called "fake superpotential" [6]. The possibilities seem endless.

A particular interesting endeavour which is tightly related to the present work is to go from $\mathcal{N}=2$ supergravity to $\mathcal{N}=4$ supergravity [5]. There are two central charges, and hence two BPS bounds similar to the one we derived here. Depending on whether none, one or both BPS bounds are satisfied, the charged black hole solutions are non-, $\frac{1}{4}$- or $\frac{1}{2}$-BPS solutions. The action for the dilaton black hole we started with can be embedded into the $\mathcal{N}=4$ theory. We find that the two possible $\frac{1}{2}$-BPS solutions we found in Section 5.1.2, related to the two possible sign choices, reappear in this larger theory as two distinct $\frac{1}{4}$-BPS solutions. In Exercise 22.20, we showed that there are *four* different choices for the signs. The other two sign choices that did not arise as BPS solutions in our work, in fact do arise as two other $\frac{1}{4}$-BPS solutions in this $\mathcal{N}=4$ supergravity. They did not appear as solutions to our Killing conditions in our treatment since we considered only two of these four supersymmetries. In the $\mathcal{N}=4$ theory, the solutions with identical sign for the charges preserve other combinations of supersymmetries that were not treated here.

7.3 Supersymmetry as cosmic censor

A neat observation [5] is a relation between the "cosmic censorship conjecture" and supersymmetry. The cosmic censorship conjecture is the hypothesis that naked singularities (*i.e.* unshielded by an event horizon) cannot be formed through gravitational collapse. In essence: they should, as far as we know, not exist in Nature. It is therefore a bit unsettling that some black hole solutions *do* have a naked singularity. One then has to look for an argument that forbids such a mathematical solution as a physical reality: this is then called a *cosmic censor*. A possible censor is supersymmetry. The supersymmetric BPS bound, in the presence of central charges, can coincide with the lower bound on the mass to avoid naked singularities. We will show this explicitly for the family of dilaton black hole solutions with charges q, p' (with opposite signs for the charges) whose extremal limit is the BPS dilaton black holes discussed in this text.

Comparing the g_{rr} components of the dilaton black hole solution and Reissner-Nördstrom solution, we derive that $(GM)^2 = |qp'|/(4\pi)^2$ in order for their $1/r^2$ terms to agree. Since our solution is an extremal one, we know that this is in fact a lower bound on the mass, such that non-extremal black hole solutions related to this solution must satisfy

$$(GM)^2 \geq |qp'|/(4\pi)^2 \,. \tag{7.5}$$

We now show that this bound agrees with the BPS bound, $M \geq |\mathcal{Z}_\infty|$, by using equation (5.25) for the central charge. The gradient flow equations showed that all solutions flow towards a minimum of $|\mathcal{Z}|$, which corresponded to the dilaton located at the horizon. Therefore, for any boundary condition, we must have $|\mathcal{Z}_\infty| \geq |\mathcal{Z}_h|$, with equality if there is no flow at all (the dilaton field lies at the fixed point, *i.e.*, starts off at spatial infinity with its value at the horizon). Hence the bound becomes

$$M^2 \geq |\mathcal{Z}_\infty|^2 \geq |\mathcal{Z}_h|^2 = \kappa^{-4}\left(2|qp'| + e^{2\phi_h}|q|^2 + e^{-2\phi_h}|p'|^2\right) \,. \tag{7.6}$$

If we now fill in the value of $e^{-2\phi_h} = |q/p'|$ and simplify, this bound indeed agrees with equation (7.5).

Therefore, if supersymmetry is present, it ensures that the mass of the black hole is large enough such that the singularity is hidden under an event horizon for observers outside of the black hole. This could be an important reason to further investigate the implications of supersymmetry for black hole solutions, and check if supersymmetry acts as a cosmic censor in other black hole solutions.

7.4 Conclusion

To conclude, the framework of matter-coupled $\mathcal{N} = 2, D = 4$ supergravity allowed us to show that charged extremal black holes are supersymmetric solutions. These black hole solutions behave as attractors and let the scalar field 'flow' from an arbitrary value at the boundary towards a fixed value on the horizon which only depends on the charges of the black hole. The terminology of 'flow' and 'attractor' arises from the fact that the dynamics of the scalar field is captured by a set of first-order differential gradient flow equations. The flow tends towards the minimal of the absolute value of the central charge, and this minimum also gives the area and hence the entropy of the black hole. Supersymmetry appears to be able to censor naked singularities, and hence turn our mathematical equations into physical realities. It is a tool that allows us to figure out the mysteries of one of the most enigmatic objects in our universe, black holes.

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