On Algebraic Geometry and the 27 Lines on Smooth Cubic Surfaces in $\mathbb{C}P^3$

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1 Introduction

As a result of its rapid development throughout much of the previous century — an intellectual achievement that is often exclusively attributed to Alexander Grothendieck — and the universality of the mathematical language in which it is written, algebraic geometry has rightly affirmed its central role in modern day mathematics. It entertains a mutually beneficial relation to a variety of other mathematical fields, such as complex analysis, topology and number theory, and it has revealed itself to be the elemental framework on which historically significant proofs to certain key theorems are built. Deligne's treatment of the *Weil conjectures* and Wiles' proof of *Fermat's Last Theorem* are notable examples of the latter. Most of the contemporary advancements in pure mathematics therefore necessitate a profound insight into the theory of algebraic geometry, and for this reason it is still a very active and blooming field in mathematics.

In its essence, algebraic geometry is concerned with the study of zero loci of ideals of polynomials. It does so through the reconciliation of a multitude of concepts that arise in commutative algebra, and the general understanding of geometric structures and spaces that has been developed throughout history. Despite the fact that present-day algebraic geometry operates on a high level of abstraction — wherein *schemes* are the basic geometric objects of study and the premises of *category theory* serve as the guiding principles for most of its advancements — any introductory text on algebraic geometry begins with the disarming but fundamental definition of an *algebraic variety*, as being the assemblage of the topological, algebraic and geometric features we can attribute to zero loci of polynomials. The primary aspiration of this thesis is to examine these features thoroughly, as well as emphasize the necessity of the study of algebraic varieties and their properties in the solving of geometric problems. The latter is done through a detailed treatment of a central problem in projective and enumerative geometry: how many lines are contained in a general smooth cubic surface S of $\mathbb{C}P^3$? We will prove that the answer is exactly 27.

Concretely, we start by introducing the notion of an algebraic variety in the explicit context of affine geometry. To this end, we present short digressions into commutative algebra and topology, so as to fully comprehend the notion of an *affine variety* as being the natural outcome of the reflections on zero sets of polynomials in these specific fields. We proceed to cover some important properties of affine varieties, and formulate a theorem that is central to the whole of algebraic geometry: *Hilbert's Nullstellensatz.*
In the subsequent section, we repeat the above exercise in the context of projective geometry: we go over the same notions, results and definitions as we did in affine algebraic geometry, and thereby state their projective counterparts. We also treat the important concept of the *dimension of a projective variety*, in view of its use for later purposes.
We follow up with a very practical introduction to functions of projective varieties, and we give two fundamental results on so-called *morphisms of varieties.* These results will prove useful in our approach to the 27-lines problem.
We finish off with a detailed treatment of the aforementioned 27-lines problem that incorporates the theory of projective varieties and morphisms of varieties, as well as some new concepts that arise in linear algebra and general projective geometry. This procedure is therefore divided into a few intermediate steps: we first prove the existence of a single line in S, after which we proceed to find the remaining lines in very specific and varied configurations to S.

The larger part of the theory that is presented in this text is based on the material introduced by Miles Reid in *Undergraduate algebraic geometry* [9], unless stated otherwise.

Notations and preliminary definitions

We briefly give the definitions of affine and projective spaces as a way of introducing the notations that are used in this text. The preliminary theory on affine and projective geometry is treated in depth in the courses *Meetkunde I* and *Meetkunde II* at KU Leuven, see [12] and [13].

Throughout this text, we generally work with a field K for which char $K \neq 2$. The set of n-tuples of elements in K is then K^n. As a set, the affine space $A^n(K)$ is simply K^n. We say that two points $p, q \in A^n(K) \setminus \{\mathbf{0}\}$ are equivalent if there exists a non-zero constant $\lambda \in K$ such that $p = \lambda q$. Using this equivalence relation, we define the projective space of dimension n over K as $KP^n := (A^{n+1}(K) \setminus \{\mathbf{0}\})/\equiv$.

2 Commutative algebra and affine varieties

At the heart of any introductory text on algebraic geometry lies the concept of an *algebraic variety*, which, loosely stated, relates to the zero locus of an ideal of polynomials in $K[X_1, \ldots, X_n]$. Whereas the ambient spaces, be it either the affine space $A^n(K)$ or the projective space KP^{n-1}, of this zero locus differ greatly in their intrinsic nature, the idea behind the concept of an algebraic variety bears similarity in the two cases: it relies on the same framework that arises from fundamental results in commutative algebra.

We therefore start this section with a brief study of these results, namely the consequences of the *Noetherian* property of polynomial rings to the aforementioned zero locus. Through the introduction of two correspondences, V and I, we then bridge from commutative algebra to the subject of affine geometry, and give a precise formulation of the *Zariski topology* along the way. This eventually allows for a precise definition of an *affine variety*. The discussion of these concepts finally culminates in the postulation of a central theorem in algebraic geometry: *Hilbert's Nullstellensatz*.

2.1 Noetherian rings

An important property of an affine variety is the fact that it can be described by a finite number of polynomials. This is because every polynomial ring $K[X_1, ..., X_n]$ over a field K is a so-called Noetherian ring, a concept we now introduce:

Definition 2.1. A Noetherian ring R is a ring for which every ideal $I \subseteq R$ is finitely generated: there exist $f_1, \ldots, f_k \in I$ such that $I = (f_1, \ldots, f_k)$.

Any field K is Noetherian, since (0) and (1) are the only ideals. Because $\mathbb{Z}$ and $K[X]$ are both principal ideal domains, they are also Noetherian rings. There are multiple ways of constructing new Noetherian rings from known ones. We first consider the quotient rings of a Noetherian ring:

Proposition 2.2. *If R is Noetherian, and $I \subseteq R$ is an ideal, then R/I is Noetherian.*

Proof. We know that for any ideal $\bar{J} \subseteq R/I$, there exists some ideal $J \subseteq R$, with $I \subseteq J$, so that $\bar{J} = J/I$. If $f_1, \ldots, f_m$ generate J, then $\bar{f}_1, \ldots, \bar{f}_m$ generate $\bar{J}$, from which the proposition follows. □

Another possibility is to consider the polynomial ring of some known Noetherian ring R. This is precisely the content of the Hilbert basis theorem, of which a proof can be found in Reid [9, p.58].

Theorem 2.3 (Hilbert basis theorem). *If R is a Noetherian ring, then $R[X]$ is also Noetherian.*

An immediate consequence of Theorem 2.3 is that by induction, any polynomial ring $K[X_1, ..., X_n]$ over a field K is Noetherian. This will prove vital in the remainder of the section.

2.2 Affine algebraic sets and the Zariski topology

From this point forward, we denote the variables of a polynomial by $x_1, \ldots, x_n$, so as to conform to the notation that is considered standard in algebraic geometry. Recall that by identifying the affine space $A^n(K)$ with K^n as a set, we recognize that the evaluation map, which evaluates some $f \in K[x_1, \ldots, x_n]$ in all the points $P = (a_1, \ldots, a_n) \in A^n(K)$, is a homomorphism of rings. Through the introduction of correspondences V and I, we now relate the commutative algebra that was discussed above to the affine geometry we pursue to study. Along the way, we find the opportunity to equip $A^n(K)$ with a topological structure.

2.2.1 The correspondence V

There exists a fundamental relationship between ideals $J \subseteq K[x_1, \ldots, x_n]$ and subsets of the affine space $A^n(K)$, which is formulated in the following correspondence:

$$V : \{J \subseteq K[x_1, \ldots, x_n] \mid J \text{ is an ideal } \} \to \{X \subseteq A^n(K)\} : J \mapsto V(J)\,, \tag{1}$$

with $V(J)$ the standard notation for the affine subset

$$V(J) = \{P \in A^n(K) \mid f(P) = 0, \ \forall f \in J\}\,. \tag{2}$$

In other words, V maps an ideal of a polynomial ring to its zeroes in the affine space $A^n(K)$, which, for this purpose, is identified with K^n as a set. If the ideal J is generated by a single function f, then we often write $V(f)$. As a follow-up to the introduction of the V-correspondence, we present the definition of an *algebraic set* in $A^n(K)$:

Definition 2.4. A subset $V \subseteq A^n(K)$ is called an algebraic set if there exists an ideal $J \subseteq K[x_1, \ldots, x_n]$ such that $V = V(J)$.

A fundamental consequence of our study of Noetherian rings is that every algebraic set in $A^n(K)$ is the zero locus of a finite number of polynomials. Indeed, since $K[x_1, \ldots, x_n]$ is a Noetherian ring, the defining ideal $J \subseteq K[x_1, \ldots, x_n]$ of some algebraic set $V(J) \subseteq A^n(K)$ is finitely generated, i.e. $J = (f_1, \ldots, f_m)$. It then holds that $P \in V(J)$ if and only if P is a zero of f_i for $i = 1, \ldots, m$, so that precisely

$$V(J) = \{P \in A^n(K) \mid f_i(P) = 0 \text{ for } i = 1, \ldots, m\}\,.$$

The algebraic sets in $A^n(K)$ exhibit some interesting topological features, that will now be discussed.

2.2.2 The Zariski topology

We first prove the following:

Proposition 2.5. *Let I, J be ideals of $K[x_1, \ldots, x_n]$, and let $\{I_\lambda \mid \lambda \in \Lambda\}$ be a set of ideals of $K[x_1, \ldots, x_n]$, indexed by λ. The correspondence V then satisfies the following properties:*

1. *If $I \subseteq J$, then $V(I) \supseteq V(J)$, and*

2. *$V(0) = A^n(K)$, $V(K[x_1, \ldots, x_n]) = \emptyset$, and*

3. $V(I \cap J) = V(I) \cup V(J)$, *and*

4. $V(\sum_{\lambda\in\Lambda} I_\lambda) = \bigcap_{\lambda\in\Lambda} V(I_\lambda)$.

Proof. 1. Since $I \subseteq J$, it is possible to write $I = (f_1, \dots, f_m)$ and $J = (f_1, \dots, f_m, \dots, f_k)$, with $k \geq m$. If $P \in V(J)$, then $f_i(P) = 0$ for $i = 1, \dots, k$, and so in particular $f_i(P) = 0$ for $i = 1, \dots, m$, meaning $P \in V(J)$.

2. Any $P \in A^n(K)$ is a zero of the zero polynomial. On the other hand, if $J = (x_1, x_1 - 1)$, then $V(J) = \emptyset$, so that by the previous, $V(K[x_1, \dots, x_n]) = \emptyset$.

3. To prove the first inclusion, assume $P \notin V(I) \cup V(J)$. Then there exist functions $f \in I$, $g \in J$ such that $f(P) \neq 0$ and $g(P) \neq 0$. Since $fg \in I \cap J$ by definition of an ideal, and since $fg(P) \neq 0$ because $K[x_1, \dots, x_n]$ is an integral domain, we have that $P \notin V(I \cap J)$. Consequently, $V(I \cap J) \subseteq V(I) \cup V(J)$.
Conversely, seen as $I \cap J \subseteq I$, we have $V(I) \subseteq V(I \cap J)$ by the first part of this proposition. The same holds for $V(J)$, from which we conclude that $V(I) \cup V(J) \subseteq V(I \cap J)$.

4. Write $I_\lambda = (f_{\lambda 1}, \dots, f_{\lambda m_\lambda})$ for every $\lambda \in \Lambda$. Then the set $G = \{f_{\lambda i} \mid \lambda \in \Lambda, i = 1, \dots, m_\lambda\}$ generates $\sum_{\lambda\in\Lambda} I_\lambda$. If $P \in V(\sum_{\lambda\in\Lambda} I_\lambda)$, then P is a zero of every $f_{\lambda i} \in G$, so that $P \in \bigcap_{\lambda\in\Lambda} V(I_\lambda)$. Conversely, if P is a zero of every $f_{\lambda i} \in I_\lambda$ for $\lambda \in \Lambda$, then $P \in V(\sum_{\lambda\in\Lambda} I_\lambda)$. This proves the equality.

□

By virtue of the above properties, we can equip $A^n(K)$ with a topological structure, namely the Zariski topology, in which the closed sets are precisely the algebraic sets. For $n = 1$, the Zariski topology coincides with the finite complement topology on $A^1(K)$. For $K = \mathbb{R}$ or $K = \mathbb{C}$ and arbitrary n, we find that the standard topologies on $\mathbb{R}$ and $\mathbb{C}$ are finer than the Zariski topologies on the respective affine spaces, since polynomial functions are continuous functions from $A^n(\mathbb{R})$ or $A^n(\mathbb{C})$ to $\mathbb{R}$ or $\mathbb{C}$ respectively. Indeed, we can write $V(J) = \bigcap_{f\in J} f^{-1}(\{0\})$, so that algebraic sets are always closed in the standard topologies on $\mathbb{R}$ and $\mathbb{C}$.

2.2.3 The correspondence I

It is possible to introduce a sort of inverse to the correspondence V, that takes a subset $X \subseteq A^n(K)$ to all polynomials in $K[x_1, \dots, x_n]$ that vanish on it, namely

$$I : \{X \subseteq A^n(K)\} \to \{J \subseteq K[x_1, \dots, x_n] \mid J \text{ is an ideal }\} : X \mapsto I(X)\,, \tag{3}$$

with

$$I(X) = \{f \in K[x_1, \dots, x_n] \mid f(P) = 0,\ \forall P \in X\}\,. \tag{4}$$

It is easy to see that $I(X)$ is an ideal of $K[x_1, \dots, x_n]$, since for every $f \in I(X)$ and $g \in K[x_1, \dots, x_n]$, fg also vanishes at the zeroes of f seen as the evaluation map is a homomorphism of rings. We also have the following properties:

Proposition 2.6. *Let $X, Y \subseteq A^n(K)$. The correspondence I then satisfies the following properties:*

1. *If $X \subseteq Y$, then $I(X) \supseteq I(Y)$, and*

2. *$X \subseteq V(I(X))$. Equality holds if and only if X is an algebraic set, and*

3. *For any ideal $J \subseteq K[x_1, \dots, x_n]$, it holds that $J \subseteq I(V(J))$.*

Proof. 1. Every function that vanishes on all points of Y naturally vanishes on all points of its subsets X, so $I(Y) \subseteq I(X)$.

2. The ideal $I(X)$ is defined as the set of all polynomials vanishing at all points of X. Then, for any point P of X, all functions of $I(X)$ vanish at P, so by definition of V, we have that $P \in V(I(X))$.
   Suppose $X = V(I(X))$. Since $I(X)$ is an ideal, X is an algebraic set by definition. Conversely, suppose that X is an algebraic set, i.e. there exists an ideal J such that $X = V(J)$. Then we know that $J \subseteq I(X)$, and by Proposition 2.5 (1), we have $V(I(X)) \subseteq V(J) = X$. Combined with the reverse inclusion that we have already proven, this results in the equality $X = V(I(X))$.

3. Every function $f \in J$ vanishes at all the points of $V(J)$ by definition, so that the inclusion $J \subseteq I(V(J))$ follows at once.

$\square$

The inclusion in Proposition 2.6 (3) can be strict: assume for instance that K is not algebraically closed, so that there exists a non-constant $f \in K[x]$ that does not have roots in K. We then have $V(f) = \emptyset$, implying that $I(V(f)) = K[x]$ trivially. But $(f) \neq K[x]$, since f is non-constant so that $1 \notin (f)$. As such, we have $(f) \subsetneq I(V(f))$.

2.3 Definition of an affine variety

We need to impose one more topological condition on the algebraic sets, i.e. the Zariski closed subsets of $A^n(K)$, to be able to give a definition of an affine variety. That condition takes the form of *irreducibility* of the algebraic set in question.

2.3.1 Irreducible algebraic sets

We hereby introduce the concept of irreducibility:

Definition 2.7. An algebraic set $X \subseteq A^n(K)$ is said to be irreducible if there does not exist a decomposition

$$X = X_1 \cup X_2, \text{ with } X_1, X_2 \subsetneq X$$

of X as a union of two strict algebraic subsets. If there exists such a decomposition, then X is said to be reducible.

Two fundamental consequences of this definition are that irreducible algebraic sets are precisely the zero loci of prime ideals, and that every algebraic set can be written as a union of irreducible components. This is the content of the Proposition 2.8. We do not prove the second statement, because it would deviate too much from the main subject of this section, but see Reid [9, p.61].

Proposition 2.8.

1. *If $X \subseteq A^n(K)$ is an algebraic set, then the following statements are equivalent:*
   - *(a) X is irreducible.*
   - *(b) I(X) is prime.*
2. *If $X \subseteq A^n(K)$ is an algebraic set, then we can write*

$$X = X_1 \cup \cdots \cup X_r \tag{5}$$

with X_i irreducible for $i = 1, \ldots, r$, and $X_i \not\subseteq X_j$ for $i \neq j$. This decomposition is also unique.

Proof. (of the first statement) It is more convenient to prove the contrapositive of the two implications. Assume therefore that X is reducible, i.e. $X = X_1 \cup X_2$ with X_1 and X_2 strict algebraic subsets of X. The strictness of the inclusion implies that there exist $f_1 \in I(X_1) \setminus I(X)$ and $f_2 \in I(X_2) \setminus I(X)$. Naturally, $f_1 f_2$ vanishes at all points of X because X is the union of X_1 and X_2. Consequently, $f_1 f_2 \in I(X)$, but $f_1, f_2 \notin I(X)$, so that $I(X)$ is not prime.
Conversely, assume that $I(X)$ is not prime. Then there exist $f_1, f_2 \notin I(X)$ with $f_1 f_2 \in I(X)$. By setting $I_1 = (I(X), f_1)$ and $X_1 = V(I_1)$, we find $X_1 \subsetneq X$, and similarly, with $I_2 = (I(X), f_2)$ and $X_2 = V(I_2)$, we have $X_2 \subsetneq X$. This means we already have $X_1 \cup X_2 \subseteq X$. The reverse inclusion is found by observing that for all $P \in X$, we have $f_1 f_2(P) = 0$, and because $K[x_1, \ldots, x_n]$ is an integral domain, we have $f_1(P) = 0$ or $f_2(P) = 0$. As such, we either need $P \in X_1$ or $P \in X_2$, so that indeed $X = X_1 \cup X_2$. □

With the introduction of the irreducibility property of algebraic sets, we now have everything at hand to give a concise definition of an affine variety.

2.3.2 Affine varieties and their properties

The following definition, when interpreted in the suitably generalized context of algebraic varieties, is fundamental in all of algebraic geometry:

Definition 2.9. An irreducible algebraic subset $V \subseteq A^n(K)$ is called an affine variety.

By Proposition 2.8 (1), we now know that affine varieties are precisely the zero loci of prime ideals in $K[x_1, \ldots, x_n]$. We also have the following equivalent properties of affine varieties:

Proposition 2.10. *Consider the Zariski topology on $A^n(K)$ and let $V \subseteq A^n(K)$ be closed. Then the following conditions on V are equivalent:*

1. *V is an affine variety.*
2. *Any two open and non-empty subsets $U_1, U_2 \subseteq V$ have $U_1 \cap U_1 \neq \emptyset$.*
3. *Any non-empty open subset $U \subseteq V$ is dense in V.*

Proof. By the definition of a dense set of a topological space, we find that the third condition is mainly a restatement of the second, since a dense set is a set that has non-empty intersection with all the opens in the topological space. As such, we only need to focus on the equivalence between the first condition and the second. Note that the condition $U_1 \cap U_2 \neq \emptyset$ for non-empty open subsets $U_1, U_2 \subseteq V$ is equivalent to the condition $V \neq (V \setminus U_1) \cup (V \setminus U_2)$. V is irreducible if and only if it is not the union of two proper closed subsets, which by the above consideration is equivalent to $U_1 \cap U_2 \neq \emptyset$. □

2.4 The Nullstellensatz

We now assert, without proof, one of the most central theorems in algebraic geometry: Hilbert's Nullstellensatz. A detailed proof of it can be found in Reid [9, p.62]. Before we are able to state the Nulstellensatz in its entirety, we need to introduce the definition of the radical of an ideal:

Definition 2.11. Let R be a ring, and $J \subseteq R$ an ideal. The radical of J is defined as the ideal

$$\text{rad } J = \sqrt{J} = \{f \in J \mid f^n \in J \text{ for some } n \in \mathbb{N}\}. \tag{6}$$

We say that an ideal J is radical if rad $J = J$. Note that the inclusion $J \subseteq$ rad J is always satisfied.

With the above definition of the radical of an ideal, and along with our knowledge of the V- and I-correspondences, we have now introduced all the concepts that are needed to formulate the Nullstellensatz:

Theorem 2.12 (Affine Nullstellensatz). *Let K be an algebraically closed field. We then have the following:*

1. *If $J \subseteq K[x_1, \ldots, x_n]$ is an ideal such that $J \neq (1)$, then $V(J) \neq \emptyset$.*
2. *$I(V(J)) = rad\ J$ for any ideal $J \subseteq K[x_1, \ldots, x_n]$.*

The Nullstellensatz shows in particular that the inclusion in Proposition 2.6 (3) is an equality if J is a radical ideal.

3 Projective varieties

The definitions and results of the previous section were all introduced in the context of the affine space $A^n(K)$. A treatment of the same concepts in projective geometry can only be carried out if we limit our study to ideals of *homogeneous polynomials*, for otherwise the traditional notions of algebraic sets and varieties would be ill-defined in KP^n. Aside from this restriction however, all of the familiar definitions and results in $A^n(K)$ can be brought over to KP^n, essentially verbatim. In this section, we aim to do just that. Additionally, we discuss the concepts of *tangent spaces* and *non-singularity*, from which point forward we can introduce the notion of the *dimension of a projective variety*. The latter will prove vital in our approach to the 27-lines problem.

3.1 Algebraic sets and varieties in KP^n

Here we introduce the necessary background theory for the study of zero loci of polynomials in KP^n. This is done through a categorical recovery of the essential results from the previous section, but now in the context of projective geometry.

3.1.1 Homogeneous ideals

We begin with the preliminary definition of a homogeneous polynomial:

Definition 3.1. A homogeneous polynomial of degree d is a polynomial in $K[x_0, \ldots, x_n]$ for which every non-zero monomial has degree d. A homogeneous polynomial is also called a form.

It immediately follows from this definition that for any homogeneous $f \in K[x_0, \ldots, x_n]$ of degree d, and any $\lambda \in K$, it holds that $f(\lambda x_0, \ldots, \lambda x_n) = \lambda^d f(x_0, \ldots, x_n)$. It is this property of homogeneous polynomials that allows for an unambiguous definition of algebraic sets in projective geometry. Another general property is that every polynomial $f \in K[x_0, \ldots, x_n]$ can be decomposed in a unique way as a sum of homogeneous polynomials, i.e. $f = \sum_{i=0}^{\deg f} f_i$, with $f_i \in K[x_0, \ldots, x_n]$ a homogeneous polynomial of degree i.

The ideals of polynomials we study in the context of projective geometry are the *homogeneous ideals*:

Definition 3.2. An ideal $J \subseteq K[x_0, \ldots, x_n]$ is said to be a homogeneous ideal if every $f \in J$ can be written as a sum of homogeneous polynomials in J, i.e. $f = \sum_{i=0}^{\deg f} f_i$, with $f_i \in J$ a homogeneous polynomial of degree i. This constraint on J is equivalent to the requirement that J is generated by homogeneous polynomials.

3.1.2 The homogeneous V- and I-correspondences

Entirely analogous to the affine case, but now with the restriction to homogeneous ideals, we can formulate the homogeneous V-correspondence for KP^n:

$$V : \{J \subseteq K[x_0, \dots, x_n] \mid J \text{ is a homogeneous ideal } \} \to \{X \subseteq KP^n\} : J \mapsto V(J)\,, \tag{7}$$

with

$$V(J) = \{[(p_0, \dots, p_n)] \in KP^n \mid f(p_0, \dots, p_n) = 0, \forall f \in J\}\,. \tag{8}$$

Since every $f \in J$ can be written as a sum of homogeneous polynomials $f_i \in J$ of degree i, the homogeneous correspondence V is well defined. Indeed, if $(x_0, \dots, x_n) \in K^{n+1}$ and $(y_0, \dots, y_n) \in K^{n+1}$ are both representatives of the same point $P \in KP^n$, then $x_i = \lambda y_i$ for some $\lambda \in K\backslash\{0\}$ and for every $i = 0, \dots, n$. If every $f \in J$ vanishes on $(x_0, \dots, x_n)$, then because $f = \sum_{i=0}^{\deg f} f_i$ and $f_i \in J$ is homogeneous, we find $f(y_0, \dots, y_n) = \sum_{i=0}^{\deg f} \lambda^i f_i(x_0, \dots, x_n) = 0$.

Definition 3.3. A subset $X \subseteq KP^n$ is called a projective algebraic set if there exists a homogeneous ideal $J \subseteq K[x_0, \dots, x_n]$ such that $X = V(J)$.

By Theorem 2.3, every homogeneous ideal in $K[x_0, \dots, x_n]$ is finitely generated. Consequently, an ideal $J \subseteq K[x_0, \dots, x_n]$ is a homogeneous ideal if and only if there exist homogeneous polynomials $f_1, \dots, f_m \in K[x_0, \dots, x_n]$ such that $J = (f_1, \dots, f_m)$. As a result, every projective algebraic set is the zero locus of a finite number of homogeneous polynomials.

The homogeneous I-correspondence for KP^n is defined as the following relation:

$$I : \{X \subseteq KP^n\} \to \{J \subseteq K[x_0, \dots, x_n] \mid J \text{ is a homogeneous ideal } \} : X \mapsto I(X)\,, \tag{9}$$

with

$$I(X) = \{f \in K[x_0, \dots, x_n] \mid \forall\ [(p_0, \dots, p_n)] \in X : f(p_0, \dots, p_n) = 0\}\,. \tag{10}$$

For every $X \subseteq KP^n$, $I(X)$ is in fact a homogeneous ideal. That is the content of the following lemma:

Lemma 3.4. *Let $X \subseteq KP^n$. Then $I(X)$ is a homogeneous ideal.*

Proof. Following the same reasoning as in the affine case, $I(X)$ is an ideal in $K[x_0, \dots, x_n]$. We still have to prove it is homogeneous as an ideal. Therefore, pick any $f \in I(X)$ and assume $\deg f = d$. Write f in its unique decomposition as a sum of homogeneous polynomials, i.e. $f = \sum f_i$, with $f_i \in K[x_0, \dots, x_n]$ a form of degree i. Choose $P = [(p_0, \dots, p_n)] \in X$ arbitrarily. Because $f(p_0, \dots, p_n) = 0$ if and only if $f(\lambda p_0, \dots, \lambda p_n) = 0$ for all $\lambda \in K \setminus \{0\}$, we also find

$$g(\lambda) = f(\lambda p_0, \dots, \lambda p_n) = \sum_{i=0}^{d} f_i(\lambda p_0, \dots, \lambda p_n) = \sum_{i=0}^{d} \lambda^i f_i(p_0, \dots, p_n) = 0\,, \tag{11}$$

for all $\lambda \in K\backslash\{0\}$. This is only possible if g is the zero polynomial. Consequently, all coefficients $f_i(x_0, \dots, x_n)$ are zero, so $f_i \in I(X)$ for all i, since P was arbitrarily chosen in X. As a result, every polynomial in $I(X)$ can be written as a sum of homogeneous polynomials in $I(X)$, so that $I(X)$ is a homogeneous ideal. □

The V- and I-correspondences satisfy the same properties as in Proposition 2.6, the proofs of which follow entirely the same reasoning as in the affine case.

3.1.3 Definition of a projective variety

The topological features of V in Proposition 2.5 remain true in KP^n, since their proofs can be carried over mutatis mutandis to the projective context. As such, it is possible to equip KP^n with a topology in the same way as we did for the affine space $A^n(K)$, i.e. the closed sets are precisely the projective algebraic sets in KP^n. This topological structure on KP^n is also called the Zariski topology. In particular, we can introduce the following definition of irreducibility of a projective algebraic set:

Definition 3.5. A projective algebraic set $X \subseteq KP^n$ is said to be *irreducible* if there does not exist a decomposition

$$X = X_1 \cup X_2, \text{ with } X_1, X_2 \subsetneq X$$

of X as a union of two strict projective algebraic subsets. If there exists such a decomposition, then X is said to be reducible.

Again, a suitable alteration of Proposition 2.8 also holds in the projective space KP^n. Analogous to the definition of an algebraic variety in $A^n(K)$, we have the following definition of a central concept in algebraic geometry, namely a projective variety:

Definition 3.6. An irreducible projective algebraic subset $V \subseteq KP^n$ is called a projective variety.

Likewise, Propositions 2.8 and 2.10 also have a projective counterpart. It is important to remark that there remains a fundamental difference in the nature of algebraic varieties in $A^n(K)$ and in KP^n, even though most of the stated properties do have an analogue in the other geometric space. To illustrate this disparity, we let $J = (x_0, \ldots, x_n)$ be a homogeneous ideal, and we consider $V(J)$ in $A^{n+1}(K)$ and in KP^n respectively. In $A^{n+1}(K)$ we have $V(J) = \{(0, \ldots, 0)\}$, whereas in KP^n we have $V(J) = \emptyset$. This ideal J is referred to as the *irrelevant ideal*, because it is often the only exception to certain propositions and theorems in projective geometry. One such theorem where the irrelevant ideal clearly demands a modification of the theory on affine varieties, is the projective counterpart of the Nullstellensatz. A proof is given by Hulek in [4, p.72]:

Theorem 3.7 (Projective Nullstellensatz)**.** *Assume that K is algebraically closed. Then the following holds:*

1. $V(J) = \emptyset$ *if and only if rad* $J \supset (x_0, \ldots, x_n)$.
2. *If* $V(J) \neq \emptyset$*, then* $I(V(J)) = \text{rad } J$.

3.2 The dimension of a projective variety

We now introduce the central notions of the tangent space to a projective variety in a point, and the non-singularity condition for a (point of a) projective variety. It is with the aid of these two concepts that we are able to introduce a tractable definition of the dimension of a projective variety, which will prove convenient in our approach to the 27-lines problem.

3.2.1 Tangent spaces and singularities

For the limited purposes of this paper, we treat the act of taking the partial derivative of a polynomial as a formal algebraic operation $\frac{\partial}{\partial x_i} : K[x_0, \ldots, x_n] \to K[x_0, \ldots, x_n]$. It maps monomials x_i^n to nx_i^{n-1}, and it satisfies the traditional properties with respect to the additive and multiplicative operations on $K[x_0, \ldots, x_n]$. Now consider the following definition:

Definition 3.8. Let $V \subseteq KP^n$ be a projective variety and $P = [(p_0, \ldots, p_n)]$ a point in V. For any homogeneous $f \in K[x_0, \ldots, x_n]$, the first order part of f at P is defined as the homogeneous polynomial

$$f_P^{(1)} = \sum_{i=0}^{n} \frac{\partial f}{\partial x_i}(P) x_i \,. \tag{12}$$

We then define the tangent space to V at P to be the projective algebraic set

$$T_P V = \bigcap_{f \in I(V)} V(f_P^{(1)}) \,. \tag{13}$$

Because $I(V)$ is finitely generated in the Noetherian ring $K[x_0, \ldots, x_n]$, the intersection in (13) reduces to a finite intersection over the generators of $I(V)$. Indeed, if $I(V) = (f_1, \ldots, f_m)$ with $f_i \in K[x_0, \ldots, x_n]$ a homogeneous polynomial, then any $f \in I(V)$ can be written as $\sum_{i=1}^m g_i \cdot f_i$, with $g_i \in K[x_0, \ldots, x_n]$. By the product rule for partial derivatives, and because $P \in V$ and $f_i \in I(V)$, we find that the first order part of f reduces to $f_P^{(1)} = \sum_{i=1}^m g_i \cdot f_{i,P}^{(1)}$. It then immediately follows that the points on which all $f_P^{(1)}$ vanish are precisely the points on which the $f_{i,P}^{(1)}$ vanish for $i = 1, \ldots, m$.

As an addendum to the above definition of the tangent space of a projective variety, we remark that its affine counterpart in $A^{n+1}(K)$ is defined by a substitution of $x_i - p_i$ for x_i in the expression for the first order part of a polynomial f in (12).

If the projective variety V in the definition of tangent space is given by a principal ideal (f) for $f \in K[x_0, \ldots, x_n]$, then (13) becomes $T_P V = V(f_P^{(1)})$. In this specific case, the definition of singularity (in a point) of V takes on a particularly simple form:

Definition 3.9. Assume $V \subseteq KP^n$ is a projective variety defined by a principal ideal (f), i.e. $V = V(f)$ with $f \in K[x_0, \ldots, x_n]$. A point $P \in V$ is said to be a singular point or singularity of V if all partial derivatives of f vanish at P, that is

$$\frac{\partial f}{\partial x_i}(P) = 0, \text{ for all } i \in \{0, \ldots, n\} \,. \tag{14}$$

If this is not the case, P is called a non-singular point. In the same way, a projective variety is said to be singular if it contains a singular point. Otherwise it is said to be non-singular or smooth.

Lastly, we remark that the zero locus of the first order part of a polynomial $f \in K[x_0, \ldots, x_n]$ is a linear equation in the variables $x_0, \ldots, x_n$. As such, $V(f_P^{(1)})$ can also be thought of as a projective subspace of KP^n, so that $T_P V$ is a subspace of KP^n as well. We then say that the dimension of the tangent space $T_P V \subseteq KP^n$, denoted $\dim T_P V$, is equal to the dimension $T_P V$ has as a subspace of KP^n. In particular, if P is a singular point of V, then $\dim T_P V = n$.

3.2.2 Definition of dimension of a projective variety

With the help of the above definitions, it is now also possible to assign a dimension to a projective variety in an unambiguous way. We formulate a key result:

Proposition 3.10. *Let $V \subseteq KP^n$ be a projective variety. For any integer r, the set*

$$S(r) = \{P \in V \mid \dim T_P V \geq r\} \tag{15}$$

is closed for the Zariski topology on KP^n.

Proof. We have to show that $S(r)$ is a projective algebraic subset of KP^n. Since the homogeneous ideal $I(V)$ is finitely generated in $K[x_0, \ldots, x_n]$, we can write $I(V) = (f_1, \ldots, f_m)$ for homogeneous $f_i \in K[x_0, \ldots, x_n]$. The tangent space to V at any point $P \in V$ is then given by

$$T_P V = \bigcap_{i=1}^{m} V(f_{i,P}^{(1)}) .$$

Now recall from linear algebra that any linear subspace of K^{n+1} defined by a system of linear equations has dimension greater than or equal to $r \in \mathbb{N}$ if and only if the coefficient matrix of the system of equations has rank less than or equal to $n - r + 1$. In this specific context, we find that $P \in S(r)$ if and only if the matrix

$$J = \left(\frac{\partial f_i}{\partial x_j}(P) \right)_{\substack{i=1,\ldots,m \\ j=0,\ldots,n}}$$

has rank less than or equal to $n - r + 1$. This in its turn is equivalent to the vanishing of every $(n - r + 2) \times (n - r + 2)$-minor of J. Thus $P \in V$ has a tangent space of dimension greater than or equal to r if and only if it is a zero of every such minor, and every such minor constitutes a homogeneous polynomial equation in $x_0, \ldots, x_n$. As a consequence, $S(r)$ is a projective algebraic set of KP^n, so that it is closed for the Zariski topology on KP^n. □

The projective algebraic set $S(r)$ is of use in the proof of the following proposition, which spells out the defining property of the dimension of a projective variety:

Proposition 3.11. *Assume $V \subseteq KP^n$ is a projective variety. Then there exists a unique integer r and a dense open subset $V_0 \subseteq V$ such that*

$$\dim T_P V = r \text{ for } P \in V_0, \quad \text{and } \dim T_P V \geq r \text{ for all } P \in V .$$

Proof. Let $r = \min_{P \in V}\{\dim T_P V\}$. Naturally, $S(r) = V$ and $S(r+1) \subsetneq V$. Now let $V_0 = S(r) \setminus S(r+1) = \{P \in V \mid \dim T_P V = r\}$, which is non-empty by the definition of r. Since V_0 is equal to $V \setminus S(r+1)$ and the latter set is closed for the Zariski topology, we find that V_0 is open in the projective variety V. Then by the projective counterpart of Proposition 2.10, we have that V_0 is dense in V, which ends the proof. □

Definition 3.12. Let $V \subseteq KP^n$ be a projective variety and let r be the integer in Proposition 3.11. We then define the dimension of V to be r, and we write $\dim V = r$.

The dimension of an affine variety can be defined analogously, but now by way of the dimension the tangent space $T_P V$ has as an affine subspace of $A^{n+1}(K)$.

4 Morphisms of projective varieties

In the previous sections we established the foundations on which algebraic geometry is built, namely the theory of algebraic varieties and their topological, algebraic and geometric properties. The natural follow-up to this basic theory is the study of *morphisms* of algebraic varieties. For the purposes of this paper, we present only the basic definitions concerning functions between projective varieties, and emphasize some specific structure-preserving properties of so-called *isomorphisms*, as these properties will become of great use to us in the proof of the 27-lines problem. Since we approach the subject of morphisms of varieties from a strictly pragmatical viewpoint, this section will not delve too deeply into the general theory.

4.1 Rational functions and the function field of a projective variety

We first present the definition of a rational function:

Definition 4.1. A rational function on a projective variety $V \subseteq KP^n$ is a partially defined function $f : V \dashrightarrow K$ such that there exist homogeneous polynomials $g, h \in K[x_0, \ldots, x_n]$ of the same degree, with $f(P) = g(P)/h(P)$ for all $P \in V$ where $h(P) \neq 0$.

Rational functions are well-defined, since $f(P)$ does not depend on the particular representative of $P \in V$ as a result of the homogeneity of g and h, and the fact that g and h have the same degree. However, there are multiple ways of writing f as a quotient of polynomials on V. We therefore introduce the following equivalence relation: two quotients of polynomials $g, h \in K[x_0, \ldots, x_n]$ and $g', h' \in K[x_0, \ldots, x_n]$ are equivalent, i.e. $g/h \equiv g'/h'$, if and only if $h'g - g'h \in I(V)$. This precisely means that the quotients g/h and g'/h' define the same rational function on V, and we have the following definition:

Definition 4.2. The function field of a projective variety $V \subseteq KP^n$, denoted by $K(V)$, is defined as the field of equivalence classes

$$K(V) = \left\{ \frac{g}{h} \;\middle|\; g, h \in K[x_0, \ldots, x_n] \text{ homogeneous with } \deg g = \deg h \text{ and } h \notin I(V) \right\} / \equiv \; . \tag{16}$$

In the definition of a rational function f on a projective variety V, we have left out the possibility that for every way of writing $f = g/h$ on V, we have $h(P) = 0$ for some fixed $P \in V$. This leads to the following definition:

Definition 4.3. For $V \subseteq KP^n$ a projective variety, $f \in K(V)$ and $P \in V$, we say that f is regular at P if there exists an expression $f = g/h$ with $g, h \in K[x_0, \ldots, x_n]$ homogeneous and of the same degree, such that $h(P) \neq 0$. The domain of definition of f, or simply domain of f, is the set

$$\operatorname{dom} f = \{P \in V \mid f \text{ is regular at } P\} \, . \tag{17}$$

A rational function f is called regular if $\operatorname{dom} f = V$.

4.2 Rational maps and morphisms

We naturally extend the definition of rational functions from V to K, to rational maps between V and the m-dimensional projective space KP^m:

Definition 4.4. Assume $V \subseteq KP^n$ is a projective variety. A rational map on V is a partially defined map $f : V \dashrightarrow KP^m$ that is given by rational functions $f_0, \ldots, f_m \in K(V)$, in a way that for all $P \in V$ we have

$$f(P) = [(f_0(P), \ldots, f_m(P))] \; . \tag{18}$$

The regularity of a rational map is then determined by the regularity of the rational functions that define it:

Definition 4.5. Let $V \subseteq KP^n$ be a projective variety. A rational map $f : V \dashrightarrow KP^m$ is said to be regular at $P \in V$ if there exists an expression $f = [(f_0, \ldots, f_m)]$ with $f_i \in K(V)$, such that f_i is regular at P for $i = 0, \ldots, m$, and such that there exists at least one $i \in \{0, \ldots, m\}$ for which $f_i(P) \neq 0$.

As before, the domain of a rational map f on V is the subset of V that contains all the points of V where f is regular. A regular rational map on V is a rational map f on V such that $\operatorname{dom} f = V$. We also remark that the rational map $f = [(f_0, \ldots, f_m)]$ on V is not uniquely determined by the rational functions $f_i \in K(V)$. Indeed, for any rational function $g \in K(V)$, we find that $[(gf_0, \ldots, gf_m)]$ defines the same rational map as f. However, the regularity and hence the domain of f do depend on the chosen expression for f: if $g \in K(V)$ vanishes at some $P \in V$, then $[(gf_0, \ldots, gf_m)]$ can never be regular at P by (2) of Definition 4.4, even though f might be a regular rational map on V. This naturally leads to the following definition of a morphism of varieties:

Definition 4.6. Let $V \subseteq KP^n$ and $W \subseteq KP^m$ be projective varieties. A morphism between V and W is a regular rational map $f : V \dashrightarrow W$.

In general, compositions of rational maps may not be well-defined. Exceptions to this rule are the *dominant* rational maps, namely rational maps $f : V \dashrightarrow W$ for which $f(\text{dom } f)$ is dense in W for the Zariski topology. A proof of this statement is found in Hartshorne [2, p.24].

Definition 4.7. Assume $V \subseteq KP^n$ and $W \subseteq KP^m$ are projective varieties. A rational map $f : V \dashrightarrow W$ is said to be birational if it has a rational inverse: there exists a rational map $g : W \dashrightarrow V$ such that $f \circ g = \text{id}_{\text{dom } g}$ and $g \circ f = \text{id}_{\text{dom } f}$. If f is also regular, then we say f is an isomorphism, and we say V and W are isomorphic as projective varieties.

It can be shown that for a general birational morphism $f : V \dashrightarrow W$, there exist open subsets $V_0 \subseteq V$ and $W_0 \subseteq W$ such that $f|_{V_0} : V_0 \to W_0$ is an isomorphism, see Reid [9, p.94].

The essential content of this section, in view of later purposes, is contained in the following two propositions. We leave the second one without proof, but see Hartshorne [2, p.27].

Proposition 4.8. *If $f : X \to Y$ is a morphism of projective varieties $X \subseteq KP^n$ and $Y \subseteq KP^m$, and X is irreducible, then $f(X)$ is also irreducible.*

Proof. Assume towards contradiction that $f(X)$ is reducible and write $f(X) = X_1 \cup X_2$ with $X_1, X_2 \subsetneq f(X)$ both algebraic sets. Then $X = f^{-1}(X_1) \cup f^{-1}(X_2)$ is a union of strict algebraic subsets of X, contradicting the assumption that X is irreducible. $\square$

Proposition 4.9. *If $f : X \to Y$ is an isomorphism of projective varieties $X \subseteq KP^n$ and $Y \subseteq KP^m$, then* $\dim X = \dim Y$.

5 The 27 lines on a smooth cubic surface

As we have already mentioned in the introduction, a particular application of the study of projective varieties and their properties is found in a comprehensive proof of the following theorem:

Theorem 5.1. *Every smooth cubic surface $S \subseteq \mathbb{C}P^3$ contains exactly 27 lines.*

We now set out to prove this theorem. Throughout this section, we work with a general smooth — that is, non-singular — and irreducible cubic surface $S \subseteq \mathbb{C}P^3$ given by a homogeneous cubic $f = f(x_0, x_1, x_2, x_3)$, and try to derive general statements concerning S. Considerable effort goes into proving that there exists at least one line $l \subseteq S$, from which point we can identify the remaining lines using geometric and combinatorial arguments, relying on the theory that was established in the previous sections. The proof of the existence of a line on S requires the notion of the Grassmannian of a vector space, along with some useful results on morphisms between projective varieties. Naturally, a short digression into these subjects is presented before the main content of this section. We commence however with the explicit case of the *Fermat cubic surface*, which aims to serve as an illustrative example to Theorem 5.1 and a convenient tool to the development of the general theory.

5.1 The Fermat cubic surface as an illustrative case

The Fermat cubic surface, or Fermat surface of degree three, is the projective variety $S = V(x_0^3 + x_1^3 + x_2^3 + x_3^3)$ in $\mathbb{C}P^3$. In this particular case, we are able to count the 27 lines on S explicitly. The argument presented here is a more detailed treatment of Lemma 11.1 in the lecture notes of Gathmann [1, p.85].

By a suitable choice of the coordinate system, we can parametrise straight lines $\ell \subseteq \mathbb{C}P^3$ by

$$\begin{cases} x_0 = a_2x_2 + a_3x_3 \\ x_1 = b_2x_2 + b_3x_3 \end{cases}, \tag{19}$$

with $a_2, a_3, b_2, b_3 \in \mathbb{C}$ and up to a permutation of the coordinates. As a result, the substitution of this parametrisation of ℓ into the homogeneous polynomial of S yields a condition on the parameters a_2, a_3, b_2 and b_3, that guarantees $\ell \subseteq S$:

$$\begin{aligned} f|_\ell &= (a_2x_2 + a_3x_3)^3 + (b_2x_2 + b_3x_3)^3 + x_2^3 + x_3^3 \\ &= (a_2^3 + b_2^3 + 1)x_2^3 + (3a_2^2a_3 + 3b_2^2b_3)x_2^2x_3 + (3a_2a_3^2 + 3b_2b_3^2)x_2x_3^2 + (a_3^3 + b_3^3 + 1)x_3^3 \equiv 0\,. \end{aligned} \tag{20}$$

Since this condition is to hold for every $(x_0, x_1, x_2, x_3) \in \ell$, we deduce the following system of equations for the parameters a_2, a_3, b_2 and b_3:

$$\begin{cases} a_2^3 + b_2^3 = -1 \\ a_3^3 + b_3^3 = -1 \\ a_2^2a_3 = -b_2^2b_3 \\ a_2a_3^2 = -b_2b_3^2 \end{cases}. \tag{21}$$

At least one of the parameters has to be zero. Indeed, assume towards contradiction that a_2, a_3, b_2 and b_3 are all non-zero. Squaring the third equation and dividing by the fourth equation then results in $a_2^3 = -b_2^3$, or $a_2^3 + b_2^3 = 0$, which contradicts the first equation. As such, we can assume that $a_2 = 0$ after a possible interchange of the parameters. This precisely determines $b_2^3 = -1$ from the first equation, so that $b_3 = 0$ from the third and fourth equation, and likewise $a_3^3 = -1$ from the second equation. Conversely, this choice of parameters a_2, a_3, b_2 and b_3 conforms to the system of equations in (21), so that indeed $\ell \subseteq S$.

As a result of the above, we find that $a_2 = b_3 = 0$, and that a_3 and b_2 are the primitive third roots of -1, i.e. $a_3 = -\omega^k$ and $b_2 = -\omega^j$ with $\omega = e^{\frac{2\pi i}{3}}$ and $j, k \in \{0, 1, 2\}$. This outcome constitutes a set of nine lines for the particular choice of parametrisation of ℓ in (19), namely

$$\begin{cases} x_0 + x_3\omega^k = 0 \\ x_1 + x_2\omega^j = 0 \end{cases}, \tag{22}$$

with $j, k \in \{0, 1, 2\}$. To account for the other permutations of the coordinates, we simply observe that the defining polynomial of the Fermat cubic is symmetric in its variables, so that the above calculation can be repeated mutatis mutandis for every such permutation, yielding the final result on the 27 lines on the Fermat surface of degree three:

$$\begin{aligned} x_0 + x_3\omega^k = x_1 + x_2\omega^j = 0, &\quad j, k \in \{0, 1, 2\} \\ x_0 + x_2\omega^k = x_3 + x_1\omega^j = 0, &\quad j, k \in \{0, 1, 2\}\,. \\ x_0 + x_1\omega^k = x_3 + x_2\omega^j = 0, &\quad j, k \in \{0, 1, 2\} \end{aligned} \tag{23}$$

5.2 Grasmannians, morphisms of varieties and the non-singularity of S

We introduce the concept of the Grassmannian of a vector space, we present a useful result on morphisms of projective varieties, and we also investigate the consequences of the non-singularity condition on S. The conclusions we can draw from this will prove vital in the validation of Theorem 5.1.

5.2.1 The Grassmannian of a vector space V

This subsection is an adaptation of the theory presented by Warner in [14]. The Grassmannian $\mathrm{Gr}(k, V)$ of the vector space V is generally defined as follows:

Definition 5.2. Let V be an n-dimensional vector space over a field K, and let $k \in \mathbb{N}$ with $k \leq n$. The Grassmannian $\mathrm{Gr}(k, V)$ is then the set of all k-dimensional linear subspaces of V.

We will often denote the Grassmannian as $\mathrm{Gr}(k, n)$, since every finite-dimensional vector space is isomorphic to any other vector space with the same dimension. A few basic examples involving vector spaces over $\mathbb{C}$ are treated here:

Examples 5.3. (i). If $k = 1$, then $\mathrm{Gr}(1, n)$ is the set of all 1-dimensional linear subspaces of some n-dimensional vector space. This precisely means that $\mathrm{Gr}(1, n)$ contains all the lines through the origin, i.e. $\mathrm{Gr}(1, n) = \mathbb{C}P^{n-1}$ as a set.

(ii). For $k = 2$ and $n = 3$, we find that $\mathrm{Gr}(2, 3)$ is the set of planes that contain the origin in any 3-dimensional vector space. For each such plane there is a unique line through the origin that is perpendicular to it, so that this characterization of planes in a 3-dimensional vector space establishes a bijective correspondence between $\mathrm{Gr}(2, 3)$ and $\mathrm{Gr}(1, 3)$. Consequently, we find $\mathrm{Gr}(2, 3) = \mathbb{C}P^2$ as a set.

It can be shown that the Grassmannian is a projective variety. We refer to Hudec [3] for a proof of this fact, seen as a detailed description of it would greatly deviate from the main content of this section. Of particular interest, in view of later purposes, is the general formula for the dimension of the Grassmannian as a projective variety. We assume $K = \mathbb{C}$, and $V = \mathbb{C}^n$ from now on.

The dimension of the Grassmannian $\mathbf{Gr}(\mathbf{k}, \mathbf{n})$

The choice of a basis $\{v_1, ..., v_n\}$ for $\mathbb{C}^n$ naturally induces a map from the space of $n \times k$-matrices over $\mathbb{C}$ without zero element, i.e. $\mathbb{C}^{n\times k} \setminus \{\mathbf{0}\}$, to the collection of all linear subspaces with dimension up to k, excluding the zero-dimensional linear subspace:

$$p : \mathbb{C}^{n\times k} \setminus \{\mathbf{0}\} \to \bigsqcup_{i=1}^{k} \mathrm{Gr}(i, n) : M = (m_{ij})_{\substack{i=1,...,n \\ j=1,...,k}} \mapsto p(M) = \mathrm{span}\{w_1, ..., w_k\}\,, \tag{24}$$

where $w_j = \sum_{i=1}^{n} m_{ij} v_i$. As a consequence, $p(M) \in \mathrm{Gr}(r, n)$ if and only if $\mathrm{rank}\ M = r$. A matrix with maximal rank — that is, a matrix of rank k in this case — is said to be of full rank.

Denote by $\mathbb{C}^{n\times k}_k$ the set of matrices in $\mathbb{C}^{n\times k}$ with full rank. By adopting the usual identification $\mathbb{C}^{n\times k} \cong \mathbb{C}^{nk}$, we find the following lemma:

Lemma 5.4. *$\mathbb{C}^{n\times k}_k$ is a Zariski open subset of $\mathbb{C}^{n\times k}$.*

Proof. The set of matrices with full rank is the complement of the set of matrices with rank strictly less than k in $\mathbb{C}^{n\times k}$. The condition that a $n \times k$-matrix has rank strictly less than k is equivalent to the requirement that all of its $k \times k$-minors vanish. As such, the set of matrices with rank strictly less than k is the zero locus of a collection of polynomials in nk variables over $\mathbb{C}$, making it Zariski closed in $\mathbb{C}^{nk} \cong \mathbb{C}^{n\times k}$. Hence its complement, i.e. $\mathbb{C}_k^{n\times k}$, is a Zariski open subset of $\mathbb{C}^{n\times k}$. □

Combined with Proposition 2.10, we also find that $\mathbb{C}_k^{n\times k}$ is dense in $\mathbb{C}^{n\times k}$. Considering the particular case $k = n$ yields the following corollary:

Corollary 5.5. *The set of all invertible $n \times n$-matrices, i.e. $GL(n,\mathbb{C})$, is dense in $\mathbb{C}^{n\times n}$ for the Zariski topology.*

The map p in (24) is clearly surjective, since any choice of basis for a linear subspace of $\mathbb{C}^n$ determines a matrix containing the coordinates of said basis elements with respect to the basis $\{v_1, ..., v_n\}$ in its columns. Moreover, as an addendum to the observation that the rank of the matrix M fixes the particular Grassmannian of which $p(M)$ is an element, we find that the restriction of p to $\mathbb{C}_k^{n\times k}$ with codomain $\mathrm{Gr}(k, n)$ is still a surjective map.

However, the map p is not necessarily injective: indeed, there are many choices for a basis of a certain k-dimensional linear subspace. Any change of basis is represented by an invertible $k\times k$-matrix and vice versa, so that we can guarantee the injectivity of p through the introduction of an equivalence relation. The following lemma makes this precise:

Lemma 5.6. *Given two matrices $A, B \in \mathbb{C}^{n\times k}$, we have that $p(A) = p(B)$ if and only if there exists an invertible matrix $C \in GL(k,\mathbb{C})$ such that $A = BC$.*

Proof. If $p(A) = p(B) = W$, then rank A = rank $B = k$, and there exist two bases for the k-dimensional subspace W, say $\mathcal{B}_1 = \{u_1, \ldots, u_k\}$ and $\mathcal{B}_2 = \{u'_1 \ldots, u'_k\}$. Therefore, there exists a $C \in \mathrm{GL}(k,\mathbb{C})$ such that $u_j = \sum c_{ij}u'_i$, which represents a change of basis from $\mathcal{B}_1$ to $\mathcal{B}_2$. We then have

$$\sum_k a_{kj}v_k = u_j = \sum_i c_{ij}u'_i = \sum_i c_{ij}\sum_k b_{ki}v_k = \sum_k\sum_i b_{ki}c_{ij}v_k\,,$$

from which we deduce that $A = BC$.

Conversely, suppose there exists a $C \in \mathrm{GL}(k,\mathbb{C})$ such that $A = BC$. Using the same notation as above, we have $u_j = \sum c_{ij}u'_i$. Hence $p(A) = p(B)$. □

Given two $n\times k$-matrices A and B, we say $A \equiv B$ if and only if there exists an invertible $k\times k$-matrix C such that $A = BC$. By virtue of Lemma 5.6, we can now introduce the well-defined bijection

$$\tilde{p} : \mathbb{C}_k^{n\times k}/\equiv\, \to \mathrm{Gr}(k, n) : [A] \mapsto p(A)\,. \tag{25}$$

If we can show $\tilde{p}$ is an isomorphism of projective varieties, then by Proposition 4.9, the dimension of the Grassmannian is equal to $nk - k^2$. Indeed, since $\mathbb{C}_k^{n\times k}$ is a dense open set in $\mathbb{C}^{n\times k}$, it has dimension nk. Similarly, the dimension of $\mathrm{GL}(k,\mathbb{C})$ as a dense open set in $\mathbb{C}^{k\times k}$ is equal to k^2. Therefore, we find that the dimension of $\mathbb{C}_k^{n\times k}/\equiv$ is equal to $nk - k^2$.

Showing that $\tilde{p}$ is indeed an isomorphism of varieties consists of extensive computations and is generally done through the use of so-called *Plücker coordinates.* Providing a detailed description of this method would deviate greatly from the main subject of this thesis. The reader is therefore referred to chapter 8 of Gathmann [1] for a rigorous proof of the fact that $\tilde{p}$ is an isomorphism of projective varieties. We simply conclude this subsection with the following result:

$$\dim \mathrm{Gr}(k, n) = k(n - k)\,. \tag{26}$$

5.2.2 A result on morphisms of varieties

We now provide the proof for a proposition that will be useful when we show that the cubic surface S contains at least one line. This proof relies on two other results that are stated in Shafarevich [11, p.68 and p.77]:

1. For $k \in \mathbb{N}$ and $Y \subseteq KP^n$ a projective variety, the set V_k defined by
$$V_k = \{y \in Y \mid \dim f^{-1}(y) \geq k\} \tag{27}$$
is closed in Y.

2. If $Y \subseteq KP^n$ is a projective variety, and $X \subseteq Y$ is a closed projective subvariety of Y with $\dim X = \dim Y$, then $X = Y$.

Proposition 5.7. *Let $f : X \to Y$ be a morphism between two projective varieties X and Y, with* $\dim X = \dim Y$*. If there exists a point $y \in Y$ such that $f^{-1}(y)$ is finite, then f is surjective.*

Proof. Note that $f(X) = V_0$ in the notation of (27), and $f(X)$ is irreducible by Proposition 4.8. We also have that $V_0 \setminus V_1$ is non-empty by the assumption that there exists a $y \in Y$ such that $f^{-1}(y)$ is finite. Since V_1 is closed, $V_0 \setminus V_1$ is open and therefore dense in V_0. From this, we conclude $\dim X = \dim V_0$, so $\dim X = \dim f(X)$, implying $\dim f(X) = \dim Y$. Moreover, $V_0 = f(X)$ is closed in Y, such that $f(X) = Y$ by the above. Hence f is surjective. □

5.2.3 Consequences of the non-singularity of S

As a result of the non-singularity condition we impose on S, we are able to prove the following proposition:

Proposition 5.8. *1. Every plane $\Pi \subseteq \mathbb{C}P^3$ intersects S in one of the following:*

(a) an irreducible cubic; or

(b) a conic plus a line; or

(c) three distinct lines.

2. As a consequence, there exist at most three lines in S through any point $P \in S$, and they are always coplanar.

Proof. 1. The only other possibility for the intersection of a plane $\Pi \subseteq \mathbb{C}P^3$ with the non-singular cubic surface S that is not listed above, is the possibility of a multiple line intersection. In other words, we have to prove that the intersection of Π with S can never be a multiple line. Assume towards contradiction that the intersection is the multiple line ℓ, and assume further that a projective transformation fixes $\Pi \leftrightarrow x_3 = 0$ and $\ell \leftrightarrow x_2 = x_3 = 0$. The assumption that ℓ is a multiple line of $S \cap \Pi$ means that f can be written as
$$f = x_2^2\alpha_1(x_0, x_1, x_2, x_3) + x_3\alpha_2(x_0, x_1, x_2, x_3)\,, \tag{28}$$
where α_1 is a linear form and α_2 is a quadratic form. Adopting this notation for f illustrates that S is certainly singular at the points for which $x_2 = x_3 = \alpha_2 = 0$. These points are the roots of α_2 on the line ℓ, which constitutes a non-empty set, as $\mathbb{C}$ is algebraically closed. Consequently, S contains singular points, contradicting the assumption that it is non-singular. As such, ℓ can not be a multiple line.

2. Any line $\ell \subseteq S$ through a point $P \in S$ is also contained in the tangent space T_PS to S at P, since $\ell = T_P\ell$ and $T_P\ell \subseteq T_PS$. This tangent space T_PS is a plane in $\mathbb{C}P^3$, so that by the above, it intersects S in at most three distinct lines. Consequently, there are no more than three lines in S through P. Moreover, they are always coplanar, as they are contained in the plane T_PS.

□

The previous proposition will prove useful in the remainder of this section. We also find the subsequent corollary:

Corollary 5.9. *Let $\Pi \subseteq \mathbb{C}P^3$ be a plane that contains three distinct lines of S. Any other line $n \subseteq S$ then intersects exactly one of these three lines.*

Proof. By the dimension theorem for projective subspaces, we find that for any line n in S, n and Π either intersect in a single point, or $n \subseteq \Pi$. The second case is impossible, since otherwise the plane Π would contain four distinct lines, contradicting Proposition 5.8 (1). In the first case, this single point needs to be contained in at least one of the three lines, otherwise it would again contradict Proposition 5.8 (1). Likewise, the point is also contained in at most one line of S, for the contrary would oppose the second part of the same proposition. Consequently, n intersects exactly one of the three lines of $\Pi \cap S$. □

5.3 The existence of a line in S

Now that all the auxiliary results have been established, we are able to focus on finding all the 27 lines in S explicitly. We begin by showing that in any cubic surface S, there is at least one line $\ell \subseteq S$.

Proposition 5.10. *Let $S \subseteq \mathbb{C}P^3$ be a cubic surface. Then there exists a line $\ell \subseteq S$.*

Proof. Recall that lines in $\mathbb{C}P^3$ admit a one-to-one correspondence to affine planes H through the origin in $A^4(\mathbb{C})$, which in their turn can be regarded as two dimensional linear subspaces in $\mathbb{C}^4$. The lines in $\mathbb{C}P^3$ are therefore parametrised by the Grassmannian $\mathrm{Gr}(2, 4)$. As a consequence, there exists a one-to-one correspondence between $\mathrm{Gr}(2, 4)$ and the lines $\ell \subseteq \mathbb{C}P^3$. By the dimension formula for the general Grassmannian $\mathrm{Gr}(k, n)$ that we have justified above, we find $\dim \mathrm{Gr}(2, 4) = 4$.

Consider the parameter space of all cubic surfaces in $\mathbb{C}P^3$, which we denote by $\mathcal{S}_3$. A cubic surface $S \subseteq \mathbb{C}P^3$ is fully determined by a homogeneous polynomial f of degree three in the homogeneous coordinates x_0, x_1, x_2, x_3. As such, f admits only a limited number of terms, because of the restrictions due to its degree and homogeneity. Indeed, by way of a combinatorial argument, we find that there are $\binom{6}{3} = 20$ third-degree monomials in the variables x_0, x_1, x_2, x_3. Hence the parameter space of all cubic surfaces has dimension 19, since we always lose one degree of freedom when we fix the coefficients of the defining polynomial f in $f = 0$. $\mathcal{S}_3$ can therefore be identified with the projective space of dimension 19, i.e. $\mathbb{C}P^{19}$.

Now define the projective variety Z to be

$$Z \equiv \{(\ell, S) \mid \ell \in \mathrm{Gr}(2, 4),\ S \in \mathbb{C}P^{19},\ \ell \subseteq S\} \subseteq \mathrm{Gr}(2, 4) \times \mathbb{C}P^{19}\,. \tag{29}$$

To determine the dimension of Z, consider the projection on the first coordinate, i.e. the morphism $\pi_1 : Z \to \mathrm{Gr}(2, 4)$, so that for a general line $\ell \in \mathrm{Gr}(2, 4)$ we have

$$\dim Z = \dim \mathrm{Gr}(2, 4) + \dim \pi_1^{-1}(\ell)\,. \tag{30}$$

We claim that the dimension of $\pi_1^{-1}(\ell)$ is equal to 15. Indeed, take $S \in \mathcal{S}_3$ such that $\ell \subseteq S$ and choose the parametrisation of the line ℓ in a way that it only contains the homogeneous coordinates x_0 and x_1. Upon writing $S = V(f)$ and using the condition $\ell \subseteq S$, we find that $f|_\ell$ satisfies

$$f|_\ell = \alpha_1 x_0^3 + \alpha_2 x_0^2 x_1 + \alpha_3 x_0 x_1^2 + \alpha_4 x_1^3 \equiv 0\,. \tag{31}$$

This is identically zero if and only if all α_i vanish. As a result, we lose four degrees of freedom for the coefficients of f. By recalling the fact that the dimension of all possible cubic surfaces is 19, we find that the dimension of $\pi_1^{-1}(\ell)$ is equal to 15. Then by (30), $\dim Z$ is equal to 19.

We explicitly showed that the Fermat cubic surface S_0 contains finitely many lines — 27 to be precise — at the beginning of this section. If we then consider the other projection morphism

$$\pi_2 : Z \to \mathbb{C}P^{19}\,, \tag{32}$$

and use the fact that Z has the same dimension as $\mathbb{C}P^{19}$ and $\pi_2^{-1}(S_0)$ is finite, then we know by Proposition 5.7 that π_2 is surjective. Hence given a cubic surface $S \subseteq \mathbb{C}P^3$, there exists a couple $(\ell, S) \in Z$ such that $\ell \subseteq S$, proving that any cubic surface contains a line ℓ. □

5.4 The remaining lines of S

By virtue of Proposition 5.10, we know that S contains at least one line ℓ. Finding the remaining 26 lines is now an undertaking on its own, that we have divided into a few intermediate steps: first we prove the existence of five pairs of lines in S that are in a specific configuration to ℓ. We follow up by finding five more lines in a similar configuration to a newfound line m as the previous ten lines are to ℓ. Finally, the remaining ten lines are found by way of a strictly geometric and combinatorial argument.

5.4.1 Finding ten more lines

As a first step in the determination of the remaining lines in S, we identify five pairs of new lines that are categorically structured with respect to the original line $\ell \subseteq S$. That is precisely the content of the following proposition:

Proposition 5.11. *Given a line $\ell \subseteq S$, there exist exactly 5 pairs (ℓ_i, ℓ_i') of lines of S that intersect ℓ. These pairs (ℓ_i, ℓ_i') are structured in a way that*

1. *for $i \in \{1, ..., 5\}$, ℓ, ℓ_i, and ℓ_i' are contained in a plane Π_i, and*
2. *for $i \neq j$, $(\ell_i \cup \ell_i') \cap (\ell_j \cup \ell_j') = \emptyset$.*

Proof. If $\Pi \subseteq \mathbb{C}P^3$ is a plane that contains ℓ, then by virtue of Proposition 5.8, the intersection $\Pi \cap S$ contains the line ℓ and a conic $\mathcal{C}$, which can be either singular or non-singular. If $\mathcal{C}$ is singular, then it is the union of two distinct lines ℓ_i and ℓ_i' that are different from ℓ, also by Proposition 5.8. We thus have to prove that there exist exactly five distinct planes Π_i for which $\mathcal{C}$ is singular. This then proves the existence of five pairs of lines (ℓ_i, ℓ_i') in S, as well as the first statement.

We can assume that $\ell \leftrightarrow x_2 = x_3 = 0$ after a suitable projective transformation. Since $\ell \subseteq S$, we are allowed to write the defining polynomial f of S as

$$f = Ax_0^2 + Bx_0x_1 + Cx_1^2 + Dx_0 + Ex_1 + F\,, \tag{33}$$

with $A, B, C, D, E, F \in \mathbb{C}[x_2, x_3]$. Because f is a homogeneous cubic polynomial over $\mathbb{C}$, we find that A, B and C are linear forms, D and E are quadratic forms, and F is a cubic form. We then observe that the singularity condition on $\mathcal{C}$ is equivalent to the condition that for every $P \in \Pi$, the homogeneous fifth degree polynomial

$$\begin{aligned}\Delta(x_2, x_3) &\equiv 4 \cdot \det \begin{vmatrix} A & B/2 & D/2 \\ B/2 & C & E/2 \\ D/2 & E/2 & F \end{vmatrix} \\ &= 4ACF + BDE - AE^2 - B^2F - CD^2\end{aligned} \tag{34}$$

vanishes at P. This is because any plane Π that contains ℓ is of the form $\Pi \leftrightarrow \mu x_2 + \gamma x_3 = 0$, with μ and γ not both equal to zero. Depending on whether μ or γ is different from zero (if they are both non-zero, then we can choose any of the two), $\mathcal{C}$ can be brought in explicit form. Take for example $\mu \neq 0$, so that we can divide both sides of the equation of Π by μ. Then $\Pi \leftrightarrow x_2 = \lambda x_3$ for some $\lambda \in \mathbb{C}$, so that $f|_\Pi = x_3 Q(x_0, x_1, x_3)$, since $\ell \subseteq S$ and with Q the equation for $\mathcal{C}$. Upon comparison with (33), we find that

$$\begin{aligned} Q(x_0, x_1, x_3) = {} & A(\lambda, 1)x_0^2 + B(\lambda, 1)x_0 x_1 + C(\lambda, 1)x_1^2 \\ & + D(\lambda, 1)x_0 x_3 + E(\lambda, 1)x_1 x_3 + F(\lambda, 1)x_3^2 . \end{aligned} \tag{35}$$

For every $P \in \Pi$, Δ reduces to $4x_3^5$ times the determinant of the defining matrix of $\mathcal{C}$ in (35), since $x_2 = \lambda x_3$ for such P. Consequently, the vanishing of Δ at every point of Π is equivalent to the singularity of $\mathcal{C}$. The case $\gamma \neq 0$ leads to an entirely similar result, with the roles of the coordinates x_2 and x_3 exchanged.

Because $\mathbb{C}$ is algebraically closed, Δ has exactly five roots, counted with their multiplicities. Since every such root corresponds to a plane Π, we have to show that Δ admits only simple roots. The projective transformation above can be adjusted so as to permit $\Pi_1 \leftrightarrow x_2 = 0$ as a root of Δ, and with $\ell \leftrightarrow x_3 = 0$, $\ell_1 \leftrightarrow x_0 = 0$ and $\ell_1' \leftrightarrow x_1 = 0$ as the lines of S in Π_1 in the case they are not concurrent, or with $\ell \leftrightarrow x_3 = 0$, $\ell_1 \leftrightarrow x_0 = 0$ and $\ell_1' \leftrightarrow x_0 = x_3$ as the lines of S in Π_1 when they are in fact concurrent. We now distinguish between these two cases:

1. **First case: the lines of $\Pi_1 \cap S$ are not concurrent.** Our aim is to prove that Δ is not divisible by x_2^2. Because $\ell, \ell_1, \ell_1' \subseteq S$, we find that $f = x_0 x_1 x_3 + x_2 g$, with g a quadratic form. Upon comparison with (33), we find that $B = x_3 + a x_2$, with $a \in \mathbb{C}$, and that $x_2 \mid A, C, D, E, F$. This precisely means that

   $$\Delta \equiv -x_3^2 F \mod x_2^2 . \tag{36}$$

   Additionally, the point $P = (0, 0, 0, 1)$ is contained in S. But the non-singularity condition of S at P then requires that F contains a term $x_2 x_3^2$, so that x_2^2 does not divide F, and $x_2^2 \nmid \Delta$. Consequently, the plane $\Pi_1 \leftrightarrow x_2 = 0$ is a simple root of Δ. This proves the first part of Proposition 5.11 in the case ℓ, ℓ_1 and ℓ_1' are not concurrent.

2. **Second case: the lines of $\Pi_1 \cap S$ are concurrent.** Again, since $\ell, \ell_1, \ell_1' \subseteq S$, we can write $f = x_0 x_3 (x_0 - x_3) + x_2 g = x_0^2 x_3 - x_0 x_3^2 + x_2 g$, with g a quadratic form. By comparing this expression for f with the expression in (33), we find $A = x_3 + a x_2$ and $D = -x_3^2 + b x_2 x_3 + c x_2^2$, with $a, b, c \in \mathbb{C}$. Furthermore, we have $x_2 \mid B, C, E, F$, so that

   $$\Delta \equiv -x_3^4 C \mod x_2^2 . \tag{37}$$

   This immediately proves that Δ is not divisible by x_2^2, since C is a linear form in x_2 and x_3. Consequently, $\Pi_1 \leftrightarrow x_2 = 0$ is also a simple root in the case ℓ, ℓ_1 and ℓ_1' are concurrent.

The second part of the proposition is proven by the simple observation that Proposition 5.8 (2) guarantees that two lines contained in distinct planes Π_i and Π_j have to be disjoint. Indeed, if this were not the case, then their intersection point would also be contained in ℓ. But ℓ and the lines of Π_i and Π_j are not coplanar, so this is impossible. This then proves the second statement. □

We also present an immediate consequence of the above proposition:

Corollary 5.12. *There exists a disjoint pair of lines (ℓ, m) in S.*

Proof. Write $\ell = \ell_1$ and $m = \ell_2$ with the notation of the above lemma. Then by virtue of that same lemma, ℓ and m are disjoint lines in S. □

5.4.2 The last lines

We call a line $n \subseteq \mathbb{C}P^3$ a transversal of a line ℓ if $\ell \cap n \neq \emptyset$. The remaining sixteen lines of S are then found with the help of the following lemmas, the first of which is stated without proof, but see Lazarus [6, p.12]:

Lemma 5.13. *For any non-singular quadric $Q \subseteq \mathbb{C}P^3$, Q contains an infinite number of lines. These lines can be divided into two disjoint sets F_1 and F_2, so that:*

1. *all lines in F_1, respectively F_2, are pairwise disjoint, and*
2. *for every $\ell_1 \in F_1$ and $\ell_2 \in F_2$ it holds that $\ell_1 \cap \ell_2 \neq \emptyset$, and*
3. *$Q = \bigcup_{\ell_i \in F_1} \ell_i = \bigcup_{\ell_i \in F_2} \ell_i$.*

Lemma 5.14. *Let $\ell_1, \ldots, \ell_4$ be disjoint lines of $S \subseteq \mathbb{C}P^3$. Then these four lines do not lie on any quadric $Q \subseteq \mathbb{C}P^3$, and they have either one or two common transversals.*

Proof. There exists a unique smooth quadric Q such that $\ell_1, \ldots, \ell_3 \subseteq Q$. Indeed, by a suitable change of coordinates, we can assume that the lines have equations $\ell_1 \leftrightarrow x_0 = x_1 = 0, \ell_2 \leftrightarrow x_2 = x_3 = 0$ and $\ell_3 \leftrightarrow x_0 = x_2, x_1 = x_3$. Then these three lines are clearly disjoint from each other, and they are all contained in the quadric $Q = V(f) = V(x_0x_3 - x_1x_2)$, since $f|_{\ell_i} = 0$ for $i = 1, \ldots, 3$, and this is the only quadric for which this is possible. However, $\ell_4 \not\subseteq Q$, since otherwise $Q \subseteq S$, which contradicts the irreducibility of S.

By Bézout's theorem for hypersurfaces in higher dimensions, and by $\ell_4 \not\subseteq Q$, we know that the intersection of ℓ_4 with Q is a finite, but non-empty set of points $\ell_4 \cap Q = \{P_1, P_2\}$, with the possibility $P_1 = P_2$. Additionally, any transversal n of $\ell_1, \ldots, \ell_3$ must be contained in Q, again by Bézout's theorem, seen as n intersects Q in at least three distinct points. The number of transversals of $\ell_1, \ldots, \ell_4$ is then the number of lines of Q through P_1 and P_2 respectively, since these also intersect ℓ_1, ℓ_2 and ℓ_3 by Lemma 5.13 (2). By the remainder of the same lemma, there is exactly one line through P_1 that is also contained in Q. Likewise, there is exactly one line of Q through P_2, with the possibility that this line coincides with the line through P_1 if either $P_1 = P_2$, or if the line through P_1 and P_2 is already contained in Q. Therefore $\ell_1, \ldots, \ell_4$ admit exactly one or two common transversals. □

We now have everything at our disposal to find the 27 lines of S and finish off the proof to Theorem 5.1. By Corollary 5.12, there exist two disjoint lines ℓ and m in S. An application of Proposition 5.11 to the existing line ℓ learns that there exist exactly five pairs (ℓ_i, ℓ_i') that intersect ℓ, in a specific configuration to ℓ as is prescribed by the proposition. Since $m \subseteq S$ is disjoint from ℓ, we know that m differs from all the ℓ_i and ℓ_i'. By Corollary 5.9 then, m intersects Π_i in either a point on ℓ_i or a point on ℓ_i' for every $i = 1, \ldots, 5$, but it never intersects both. We can then renumber the pairs (ℓ_i, ℓ_i') so that m meets ℓ_i for $i = 1, \ldots, 5$.

Another application of Proposition 5.11, now to m, shows that there exist exactly five more distinct lines $\ell_i'' \subseteq S$ that intersect m, since the lines ℓ_i are already accounted for. These ℓ_i'' also differ from the ℓ_i', since the former intersect m and the latter do not. To sum it up, we now have two disjoint lines, ℓ and m, and five triples $(\ell_i, \ell_i', \ell_i'')$ so that ℓ is coplanar with ℓ_i and ℓ_i' for $i = 1, \ldots, 5$, and m is coplanar with ℓ_i and ℓ_i'' for $i = 1, \ldots, 5$. We have already found 17 lines at this point.

As for the configuration of the ℓ_i'' with respect to the ℓ_j and ℓ_j' for $i \neq j$, we observe the following: by Proposition 5.11 (2) on m, $\ell_i'' \cap \ell_j = \emptyset$ for $i \neq j$. On the other hand, ℓ_i'' and ℓ_j' have non-empty intersection for $i \neq j$. Indeed, by Corollary 5.9, the line ℓ_i'' intersects Π_j in either ℓ, ℓ_j or ℓ'_j. But ℓ''_i can not intersect ℓ since ℓ''_i differs from the ℓ_i and ℓ'_i, which are the only lines of S intersecting ℓ, and ℓ''_i also does not intersect ℓ_j for $i \neq j$ by the argument before. Hence ℓ''_i intersects ℓ'_j for $i \neq j$.

The remaining ten lines are then found with the help of this last proposition:

Proposition 5.15. *With the configuration of the lines ℓ, m and $(\ell_i, \ell'_i, \ell''_i)$ for $i = 1, \ldots, 5$ as established above, we have the following:*

1. *If $n \subseteq S$ is any other line than the seventeen above, then n meets exactly three out of the five lines $\ell_1, \ldots, \ell_5$.*
2. *Conversely, given any choice $\{i, j, k\}$ of three distinct elements of the set $\{1, 2, 3, 4, 5\}$, there exists a unique line $\ell_{ijk} \subseteq S$ meeting ℓ_i, ℓ_j and ℓ_k that is different from the seventeen lines above.*

Proof. 1. The line $n \subseteq S$ can not meet four lines out of $\ell_1, \ldots, \ell_5$, since n would then be a common transversal of these four lines, of which we know ℓ and m already are common transversals. Then by Lemma 5.14, either $n = \ell$ or $n = m$, and both considerations contradict the assumption that n differs from the already established lines.

   On the other hand, n can not meet two lines or less out of $\ell_1, \ldots, \ell_5$. Indeed, assume towards contradiction that it does. By Corollary 5.9 and Proposition 5.11, for every $i = 1, \ldots, 5$, n either meets ℓ_i or ℓ'_i, although never both. Consequently, if n meets two or less of the ℓ_i, it must meet three or more of the ℓ'_i, namely these with the complementary indices. By a renumbering of the $(\ell_i, \ell'_i, \ell''_i)$-triples, we can assume that n either meets $\ell'_1, \ldots, \ell'_5$, or n meets $\ell_1, \ell'_2, \ldots, \ell'_5$, or n meets $\ell_1, \ell_2, \ell'_3, \ell'_4, \ell'_5$. But we have already illustrated that ℓ and ℓ''_1 are common transversals of $\ell_1, \ell'_2, \ell'_3, \ell'_4$ and ℓ'_5. In either of the above three cases, n meets at least four of these latter five lines, so that by Lemma 5.14, $n = \ell$ or $n = \ell''_1$. Both equalities contradict the assumption that n is different from the lines that were already constructed, so n can not meet two lines or less out of $\ell_1, \ldots, \ell_5$. By combining with what we already found above, n must meet exactly three out of the five lines $\ell_1, \ldots, \ell_5$.

2. An application of Proposition 5.11 to the line ℓ_i learns that there are exactly ten lines meeting ℓ_i. The lines ℓ, ℓ'_i, m and ℓ''_i are already accounted for, and the remaining six lines differ from the ℓ_j, ℓ'_j and ℓ''_j for $i \neq j$ by Proposition 5.11 (2) and the description of the configuration of the ℓ''_j with respect to the ℓ_i above. As a consequence of the first statement of this proposition, each of the remaining six lines must meet exactly two lines out of the set $\{\ell_j \mid j = 1, \ldots, 5, \text{ and } j \neq i\}$. There are $\binom{4}{2} = 6$ possibilities for this to happen, and so the six lines that result from this, call them ℓ_{ijk}, must form the remaining six lines that intersect ℓ_i.

□

Proposition 5.15 (2) guarantees that there exist at least $\binom{5}{3} = 10$ more lines ℓ_{ijk} on top of the seventeen lines that were already constructed before, with the unique property of intersecting ℓ_i, ℓ_j and ℓ_k. By the first part of the same proposition, we now know that these ten lines are the only lines that remained to be found in S, so that S contains exactly 27 lines. This finally ends the proof of Theorem 5.1.

6 Outlook and conclusion

In this last section, we discuss multiple subjects that are related to the theory presented in this thesis, and that can be considered as logical continuations to the introductory algebraic geometry that this text offers. We also formulate a final conclusion on this specific study of algebraic geometry.

Generalizing the foundations of algebraic geometry

Throughout the larger part of this text, we have built the theory of algebraic geometry by working over arbitrary fields, and only the last section applied this theory to the specific case of the algebraically closed field of the complex numbers. There are various reasons why this generality of the theory is to our advantage, and why it is meaningful to consider non-algebraically closed fields as well. As a way of motivating this assertion, we can look at the modus operandi in algebraic number theory, wherein the framework of algebraic geometry over the field $\mathbb{Q}$ is used to solve an abundance of problems. A famous example that we have already mentioned in the introduction is *Fermat's Last Theorem*: integer solutions to the defining equation $x^n + y^n = z^n$ for $n \geq 3$ correspond to rational points on the variety $V(x_0^n + x_1^n - x_2^n)$ in $\mathbb{C}P^2$. Other possible generalisations constitute defining an algebraic variety in a more abstract manner, which does not necessarily admit a representation in an affine or projective space, or to omit the requirement that varieties should be irreducible [2, p. 55-59].

These generalisations are all part of the approach to algebraic geometry that was introduced by Alexander Grothendieck in the previous century. The observation that affine varieties correspond to finitely generated integral domains over a field, led Grothendieck to consider varieties over general abelian rings. This contemplation then famously culminated in the introduction of schemes, and is often pointed out to be a revolution in the study of algebraic geometry in the previous century. The theory of schemes is intricately related to various other branches of mathematics, such as Galois theory and commutative algebra, and due to its technical depth and reach, it is currently considered to be the most universal foundation for algebraic geometry [5]. As another motivation for studying schemes as the central algebraic geometric objects, we point out that our treatment of the 27-lines problem can be greatly reduced in length if we adopt the results that stem from this abstracted theory, see for example Hartshorne's one page long proof of the same problem in [2, p.402].

Enumerative geometry

In the last section, we showed that there are exactly 27 lines contained in a smooth cubic surface in $\mathbb{C}P^3$. This is an example of a general problem that arises in enumerative geometry, a particular branch of algebraic geometry concerned with counting the number of solutions to certain geometric problems. We hereby present several options for a possible continuation of our study in the field of enumerative geometry.

A first possibility is to drop the restriction that the cubic surface should be smooth. If, for example, the cubic only contains *isolated singularities*, then one can show that the surface contains at least one line, but that the total number of lines is strictly less than 27, as is asserted in Pannekoek [7].

Our extensive discussion of the lines in a smooth cubic surface naturally leads to the question if similar results hold for smooth surfaces in $\mathbb{C}P^3$ that are given by polynomials with a higher degree. For instance, we can consider smooth surfaces of which the defining polynomial has degree four or five, conveniently named *quartics* and *quintics* respectively. Beniamino Segre has proven in [10] that there is always maximum of 64 lines contained in a quartic surface, and this upper bound is sharp, since it is generally known that the Schur quartic admits exactly 64 lines, see Rams [8]. It has also been proven that the maximum number of lines contained in a smooth quintic is at most 127, but whether this is a sharp bound or not is yet unproven. As such, a possible continuation of our study in enumerative geometry could constitute research of these upper bounds and their underlying theory.

Conclusion

In this thesis, we have demonstrated the extensive reach of algebraic geometry in mathematics, and that it is intricately related to algebra and topology. In particular, we have illustrated, with the help of the notions of algebraic varieties and their algebraic and topological structure, that a profound understanding of algebra and topology allows for an accurate description of central concepts in geometry, and vice versa. We have shown the potential of algebraic geometry as a framework for projective geometry by way of a tractable proof of the fact that all smooth cubic surfaces contain exactly 27 lines, and we have given the reader and ourselves an incentive to probe further into the matter of algebraic geometry.

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