On Quantum Harmonic Oscillators, Modularity and Algebraic Number Theory

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Abstract

We study the canonical partition function of a countably infinite collection of independent quantum harmonic oscillators. This partition function appears to be closely related to a complex function introduced by Richard Dedekind, the Dedekind eta function η. We show that η exhibits highly symmetric and modular properties that can then be understood from the much broader context of modular forms. Due to their periodicity, modular forms admit a Fourier expansion, from which we recover general theorems concerning the asymptotic behaviour of their Fourier coefficients. The central result of this paper constitutes the fact that the Fourier coefficient α_n of the partition function, obtained through the foray into modular forms, is exactly the degeneracy factor or number of states corresponding to a certain eigenvalue of the energy spectrum. We conclude with a consideration of the applicability of the system of an infinite amount of quantum harmonic oscillators to various other fields of physics. In particular, the connection to a two dimensional conformal field theory in the form of bosonic movement on a 2-torus is motivated, and the relation of our results to the premises and axioms of quantum field theory is made clear.

We doen onderzoek naar de canonische partitiefunctie van een aftelbaar oneindige verzameling onafhankelijke harmonische oscillatoren in de context van kwantummechanica. Deze partitiefunctie blijkt nauw verband te houden met een complexe functie geïntroduceerd door Richard Dedekind, de Dedekind eta functie η. We tonen aan dat η sterk symmetrische en modulaire eigenschappen vertoont, dewelke begrepen kunnen worden vanuit de veel algemenere context van modulaire vormen. Geholpen door hun periodisch karakter kunnen modulaire vormen ontwikkeld worden in een Fourierreeks, van waaruit we algemene stellingen halen die het asymptotische gedrag van de Fouriercoëfficiënten weergeven. Het centrale resultaat van deze thesis handelt over het feit dat de Fouriercoëfficiënt α_n van de partitiefunctie, verkregen door de studie van modulaire vormen, gelijk is aan de ontaardingsfactor ofwel het aantal toestanden dat overeenkomt met een bepaalde eigenwaarde uit het energiespectrum. We besluiten met een beschouwing van de toepasbaarheid van het systeem van een oneindig aantal harmonische oscillatoren in andere domeinen van de fysica. In het bijzonder motiveren we het verband met een tweedimensionale conforme veldentheorie in de vorm van de beweging van een boson op een 2-torus, en brengen we de verhouding tussen onze resultaten en de aannames en axioma's van kwantumveldentheorie aan het licht.

Contents

I Introduction

The quantum harmonic oscillator lies at the heart of any theory in modern physics, as it forms the guiding principle in both quantum field theory and string theory. Quantum field theory is built upon the premise that particles can be treated as excited states of their underlying fields, of which the behaviour can be studied using raising and lowering operators analogous to the ladder operators acting on the eigenstates of the quantum harmonic oscillator. As such, it is only natural that the theory of quantum harmonic oscillators is used extensively in the study of these field theories. String theory on the other hand assumes particles can be treated like strings, of which the different vibrational states determine the physical properties that the particle exhibits. The consideration of these vibrational modes necessitates a profound insight into the working of the quantum harmonic oscillator, again motivating its distinct applicability in modern physics.
An additional aspiration of this paper is to demonstrate the influence of mathematics in contemporary physics. Theoretical physics is a study of nature in its most fundamental appearance, which is done by implementing a mathematically advanced and rigorous framework. To understand various phenomena in physics, we desire a well established connection between mathematics and the structure of nature, of which the theory in this paper aspires to be an illustration.

We consider the particular system of a countably infinite amount of independent quantum harmonic oscillators, and try to distill concrete results concerning physical behaviour through study of its canonical partition function $\mathcal{Z}$. On grounds of results stemming from statistical mechanics, it is clear that much of the physics of this system is contained in its partition function. Progress in the study of this function can be made by recognizing the fact that the obtained expression for $\mathcal{Z}$ resembles another function, up to a complex substitution: the Dedekind eta function η, introduced by Richard Dedekind in 1877. As this mathematical object has already been studied to an extensive degree in a variety of contexts throughout its history, the aim of this paper is to examine this function and recover important theorems about it. These theorems in turn unveil crucial properties of $\mathcal{Z}$ and the system we are considering.
Concretely, upon investigation of certain symmetries that the Dedekind function exhibits, we find that it can be placed in the more general context of modular forms, of which we set out to provide a self-contained theoretical description. The symmetries apparent in modular forms encompass periodicity, and as such allow for a general method of Fourier expanding them. The Fourier coefficients $d(n)$ of η contain a lot of physical information about the system of quantum harmonic oscillators, since they can be considered as representations of the number of microstates corresponding to an energy E_n, and an expression for the entropy S can be derived. We in particular look for results on the asymptotics of these coefficients, as the consequential asymptotic behaviour of the entropy can then be compared to the entropy of other physical systems in the literature.
We conclude by providing different ways of extending this foray into the theory of quantum harmonic oscillators, the Dedekind function and modular forms to different domains of physics and mathematics. In particular, we motivate an understanding of the investigated concepts in the context of quantum field theory, conformal field theory, and the generalised theory of modular forms. This paper does not aim to provide a full theoretical description of these connections, as they merely serve as an incentive for the reader to probe further into the matter.

II The quantum harmonic oscillator

The physical system of a single, one-dimensional, linear quantum harmonic oscillator is described by a Hamiltonian operator

$$H = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2\,, \tag{II.1}$$

where $\omega = (k/m)^{1/2}$ denotes the angular frequency of the harmonic oscillator. In the light of our present discussion regarding the canonical partition function of an infinite amount of such harmonic oscillators, we first need to look for the eigenvalues of this Hamiltonian. After this we construct the partition function of a single harmonic oscillator.

II A. Eigenvalues of the quantum harmonic oscillator

The eigenvalues of the operator (II.1) can be determined by way of the ladder operator method proposed by Dirac, which makes no reference to any particular position or momentum representation of the eigenstates. To this end, the so-called ladder operators are introduced:

$$\begin{aligned} a &= \frac{1}{\sqrt{2}}\left[\left(\frac{m\omega}{\hbar}\right)^{1/2}\hat{x} + i\frac{\hat{p}}{(m\hbar\omega)^{1/2}}\right] \\ a^\dagger &= \frac{1}{\sqrt{2}}\left[\left(\frac{m\omega}{\hbar}\right)^{1/2}\hat{x} - i\frac{\hat{p}}{(m\hbar\omega)^{1/2}}\right]\,. \end{aligned} \tag{II.2}$$

Upon declaring $N = a^\dagger a$, where N is conveniently called the number operator, it can be shown that the Hamiltonian becomes

$$H = \left(N + \frac{1}{2}\right)\hbar\omega\,. \tag{II.3}$$

Together with the commutation relation $\left[a, a^\dagger\right] = 1$, Dirac determined the eigenvalues to be

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega\,, \tag{II.4}$$

for $n \in \mathbb{N} \cup \{0\}$. A complete description and proof of Dirac's method, adopted from [1], is provided in appendix A.

II B. The partition function of a single quantum harmonic oscillator

Using the expression for the energy eigenvalues in (II.4), we can compute the canonical partition function

$$\mathcal{Z} = \mathrm{Tr}\left(e^{-H/k_BT}\right)\,, \tag{II.5}$$

for a system of a single quantum harmonic oscillator in contact with a thermal bath at temperature T. Since the trace of a matrix is independent of choice of basis, and since the set of eigenstates $\{|\Psi_n\rangle\}$ corresponding to the eigenvalues E_n of the Hamiltonian operator comprises a complete set of eigenfunctions, this partition function is readily calculated to be

$$\begin{aligned} \mathcal{Z} &= \sum_{n=0}^{+\infty} \langle\Psi_n|\, e^{-\beta H}\, |\Psi_n\rangle \\ &= e^{-\frac{1}{2}\beta\hbar\omega}\sum_{n=0}^{+\infty}\left(e^{-\beta\hbar\omega}\right)^n\,, \end{aligned} \tag{II.6}$$

where we have written $\beta = \frac{1}{k_B T}$. This latter expression consists of a geometric series, which can be written in a closed form provided that $\left|e^{-\beta\hbar\omega}\right| < 1$. As $\beta\hbar\omega > 0$ is always true and $|e^{-x}| < 1$ for $x > 0$, this condition is at once fulfilled for every temperature T of the thermal bath. We finally get

$$\mathcal{Z} = \frac{e^{-\frac{1}{2}\beta\hbar\omega}}{1 - e^{-\beta\hbar\omega}} . \tag{II.7}$$

The numerator in this equation being equal to $e^{-\frac{1}{2}\beta\hbar\omega}$ conveniently reflects the fact that the zero-point energy of the quantum harmonic oscillator is equal to $\frac{1}{2}\hbar\omega$ instead of zero, so that even in its ground state the oscillator admits some energy. This constitutes a stark contrast to the situation in classical mechanics, for which the harmonic oscillator attains its lowest energy at $E = 0$. Lastly, we remark that it is the expression for the canonical partition function in (II.7) that will prove vital in our study of a system of an infinite amount of quantum harmonic oscillators.

III The Dedekind eta function

We now turn our attention to the system of an infinite amount of quantum harmonic oscillators. More precisely, the object of study is a countably infinite collection of independent quantum harmonic oscillators, each of which is characterized by a frequency $\omega_k = \frac{k}{\hbar}$, for $k \in \mathbb{N}$. Making use of the factorisation property of the canonical partition function for independent particles

$$\mathcal{Z} = \prod_{k=1}^{+\infty} \mathcal{Z}_k , \tag{III.1}$$

we will show that the resulting expression for $\mathcal{Z}$ can be related to a complex-valued function introduced by Richard Dedekind in 1877, the Dedekind eta function η. We then proceed to study particular transformation formulae of this function, and try to understand these distinct transformations as well as their effect on η in the much broader context of *Möbius transformations* of the upper half of the complex plane.

III A. Getting to the Dedekind eta function

With the characterization of the frequencies of the different harmonic oscillators introduced above, we note that the partition function for a quantum harmonic oscillator with such a frequency $\omega_k = \frac{k}{\hbar}$ for $k \in \mathbb{N}$ yields

$$\mathcal{Z}_k(\beta) = \frac{e^{-\frac{1}{2}\beta k}}{1 - e^{-\beta k}} . \tag{III.2}$$

If we substitute this expression into (III.1) and bring out the numerator, we get

$$\mathcal{Z}(\beta) = e^{-\frac{1}{2}\beta \sum\limits_{k=1}^{+\infty} k} \prod_{k=1}^{+\infty} \frac{1}{1 - e^{-\beta k}} . \tag{III.3}$$

On part of the well-known equality $\sum_{k=1}^{+\infty} k = \frac{-1}{12}$, a result that stems from the theory of analytic continuation of the *Riemann zeta function* and that we prove and make sensical in the appendix, we find

$$\mathcal{Z}(\beta) = e^{\frac{\beta}{24}} \prod_{k=1}^{+\infty} \frac{1}{1 - e^{-\beta k}} . \tag{III.4}$$

Having arrived at this point in the computation of the partition function, it is a good idea to elaborate a bit on the seemingly arbitrary complex substitution that will follow in order to make the connection between $\mathcal{Z}$ and the Dedekind eta function η. Up until now, $\mathcal{Z}$ has been a real function of a real variable, β, which corresponds to the fact that the temperature T is indeed a real quantity. The Dedekind eta function, however, is a complex-valued function defined on (part of) the complex plane. So as to be able to recover important results concerning this function η, a substitution on β to complex variables has to be made. Nonetheless, in the end only the real variables of β are of importance to describe the true physical nature of the system we are considering, reflecting the fact that the mere real values of the temperature T are of interest. As such, the situation is analogous to the widely used complexification of real sine waves to complex functions of the form e^{ikx}. This complexification allows for ease of computation and for the recovery of important results from complex analysis, but in the end it is necessary to take the real part of the resulting complex function in order to obtain meaningful physical results about the wave.

Concretely, we perform a substitution of the following form: write $\beta = -2\pi i\tau$, with τ a priori a complex number. Recalling the fact that each $\mathcal{Z}_k$ can be written in the closed form expression of the geometric series under the condition that $\left|e^{-\beta k}\right| < 1$ in the case of a single quantum harmonic oscillator, we can now construct a similar restriction on τ. Concretely, substituting β for $-2\pi i\tau$ in the latter inequality yields the restriction that $e^{-2\pi k \mathrm{Im}(\tau)} < 1$. Since every $k \in \mathbb{N}$ is strictly positive, this inequality boils down to the constraint that $\mathrm{Im}(\tau) > 0$, i.e. that $\tau \in \mathbb{H}$, where

$$\mathbb{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}\,(\tau) > 0\} \tag{III.5}$$

is the complex upper half-plane. The resulting partition function is then

$$\mathcal{Z}(\tau) = e^{-\frac{2\pi i\tau}{24}} \prod_{k=1}^{+\infty} \frac{1}{1 - e^{2\pi i\tau k}}\,, \tag{III.6}$$

for $\tau \in \mathbb{H}$. This is precisely the inverse of the Dedekind eta function η, which is defined as

$$\eta\colon \mathbb{H} \to \mathbb{C}\colon \tau \mapsto \eta(\tau) = e^{\frac{2\pi i\tau}{24}} \prod_{k=1}^{+\infty} \left(1 - e^{2\pi i\tau k}\right)\,. \tag{III.7}$$

We thus obtain the fundamental relation

$$\mathcal{Z}(\tau) = \frac{1}{\eta\,(\tau)} \tag{III.8}$$

for every $\tau \in \mathbb{H}$. We conclude this paragraph by rewriting the Dedekind eta function in a more workable form: define $q = e^{2\pi i\tau}$ and $D_1 = \{q \in \mathbb{C} \mid |q| < 1\}$, then (III.7) becomes

$$\eta\colon D_1 \to \mathbb{C}\colon q \mapsto \eta(q) = q^{\frac{1}{24}} \prod_{k=1}^{+\infty} \left(1 - q^k\right)\,. \tag{III.9}$$

The substitution of τ for q in the definition of the Dedekind function constitutes a map from $\mathbb{H}$ to D_1, i.e. a map from the complex upper half-plane to the open unit disk.

III B. Transformation formulae for the Dedekind function

It turns out that the Dedekind eta function satisfies two fundamental functional equations. This statement is precisely the subject of the following theorem:

Theorem III.1. *For every $\tau \in \mathbb{H}$, the following equalities hold:*

$$\begin{aligned} &(1) \quad \eta(\tau+1) = e^{\frac{\pi i}{12}}\eta(\tau) \\ &(2) \quad \eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\eta(\tau)\,. \end{aligned} \tag{III.10}$$

Proof. (1) The first property is readily verified from the definition of the Dedekind function. We directly calculate $\eta(\tau+1)$ to be

$$\begin{aligned} \eta(\tau+1) &= e^{\frac{2\pi i(\tau+1)}{24}} \prod_{k=1}^{+\infty}(1-e^{2\pi i(\tau+1)k}) \\ &= e^{\frac{\pi i}{12}} \cdot e^{\frac{2\pi i\tau}{24}} \prod_{k=1}^{+\infty}(1-e^{2\pi ik}\cdot e^{2\pi i\tau k})\,. \end{aligned}$$

Since $e^{2\pi ik} = 1$ for every $k \in \mathbb{Z}$, we get that

$$\eta(\tau+1) = e^{\frac{\pi i}{12}}\eta(\tau)\,.$$

(2) The proof of the second property is of somewhat greater complexity. We make use of the pentagonal number theorem and the Poisson summation formula, for completeness reformulated here.

The pentagonal number theorem states that the equality

$$\prod_{n=1}^{+\infty}(1-x^n) = \sum_{-\infty}^{+\infty}(-1)^k\, x^{k(3k-1)/2} \tag{III.11}$$

holds for $|x| < 1$ in the context of convergent power series.

The Poisson summation formula establishes a relation between certain values of an appropriate function f and corresponding values of the Fourier transform of the same function. Concretely, we have the equality

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{k\in\mathbb{Z}} \hat{f}(k)\,, \tag{III.12}$$

where $\hat{f}$ denotes the Fourier transform of f.

We can now proceed with the proof of (2). A change of variables $r = e^{-\frac{2\pi i}{\tau}}$ in (III.7) for evaluation of η in $\frac{-1}{\tau}$ yields

$$\eta(-1/\tau) = r^{\frac{1}{24}} \prod_{k=1}^{+\infty}(1-r^k)\,. \tag{III.13}$$

Apply the pentagonal number theorem to the right hand side of this expression:

$$\eta(-1/\tau) = r^{\frac{1}{24}} \sum_{k\in\mathbb{Z}}(-1)^k r^{\left(3k^2-k\right)/2}\,. \tag{III.14}$$

This equality is justified by virtue of the fact that $|r| < 1$ for $r = e^{-\frac{2\pi i}{\tau}}$ and $\tau \in \mathbb{H}$. Define the function f to be

$$f : \mathbb{R} \to \mathbb{R} : x \mapsto \exp\left[\pi i\left(-\frac{1}{12\tau} + x - \frac{(3x^2-x)}{\tau}\right)\right]\,, \tag{III.15}$$

such that from equation (III.14) we find $\eta(-1/\tau) = \sum_{n\in\mathbb{Z}} f(n)$. To apply Poisson's summation formula to this function, we have to compute the Fourier transform of f. Direct calculation of the Fourier transform from its definition yields

$$\hat{f}(k) = \int_{-\infty}^{+\infty} dx \exp\left[\pi i\left(-\frac{1}{12\tau} + x - 2kx - \frac{(3x^2 - x)}{\tau}\right)\right]. \tag{III.16}$$

This is a Gaussian integral of the form

$$\int_{-\infty}^{+\infty} dx \exp\bigl(-ax^2 + bx + c\bigr) = \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2}{4a} + c\right), \tag{III.17}$$

the result of which is only valid provided that $\mathrm{Re}(a) > 0$. This translates to the condition that $\mathrm{Re}\left(\frac{3\pi i}{\tau}\right) = \mathrm{Im}(-\frac{3\pi}{\tau}) > 0$, which is at once satisfied because of the restriction that $\mathrm{Im}(\tau) > 0$. In this particular case we have $a = \frac{3\pi i}{\tau}$, $b = \pi i\left(\frac{1}{\tau} + 1 - 2k\right)$ and $c = \frac{-\pi i}{12\tau}$. Substituting these values of a, b and c into (III.17), we find

$$\hat{f}(k) = \sqrt{\frac{-i\tau}{3}} \exp\left[\pi i\left(\frac{\tau(2k-1)^2}{12} - \frac{(2k-1)}{6}\right)\right]. \tag{III.18}$$

We are now able to apply Poisson resummation to this latter expression for $\hat{f}$. Doing so results in

$$\eta\left(-\frac{1}{\tau}\right) = \sqrt{\frac{-i\tau}{3}} \sum_{k\in\mathbb{Z}} \exp\left[\pi i\left(\frac{\tau(2k-1)^2}{12} - \frac{(2k-1)}{6}\right)\right]. \tag{III.19}$$

We now turn our attention to the right hand side of the transformation formula (2). Applying the pentagonal number theorem to $\eta(\tau)$ yields

$$\eta(\tau) = q^{\frac{1}{24}} \sum_{k\in\mathbb{Z}} (-1)^k q^{(3k^2-k)/2} = \sum_{n\in\mathbb{Z}} \exp\left[\pi i\left(\frac{\tau(6n-1)^2}{12} + n\right)\right]. \tag{III.20}$$

The equality formulated in (2) is established by splitting the sum in (III.19) into three sums over $k = 3l$, $k = 3l+1$ and $k = 3l+2$ for $l \in \mathbb{Z}$:

$$\begin{aligned}\eta\left(-\frac{1}{\tau}\right) = \sqrt{-\frac{i\tau}{3}} \sum_{l\in\mathbb{Z}} \Biggl(&\exp\left[\pi i\left(\frac{\tau(6l-1)^2}{12} - \frac{6l-1}{6}\right)\right] \\ &+ \exp\left[\pi i\left(\frac{\tau(6l+1)^2}{12} - \frac{6l+1}{6}\right)\right] \\ &+ \exp\left[\pi i\left(\frac{\tau(6l+3)^2}{12} - \frac{6l+3}{6}\right)\right]\Biggr).\end{aligned} \tag{III.21}$$

In the first sum, bring out a factor $\exp(\pi i/6)$. In the second sum, apply the transformation $l \to -l$ and bring out a factor $\exp(-\pi i/6)$. The two sums that remain are identical, and the factor in front of their combined sum is $\exp(\pi i/6) + \exp(-\pi i/6) = 2\cos(\pi/6) = \sqrt{3}$. These operations result in (III.21) being equal to

$$\begin{aligned}\eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau} \sum_{l\in\mathbb{Z}} \Biggl(&\exp\left[\pi i\left(\frac{\tau(6l-1)^2}{12} + l\right)\right] \\ &+ \frac{1}{\sqrt{3}} \exp\left[\pi i\left(\frac{\tau(6l+3)^2}{12} - \frac{(6l+3)}{6}\right)\right]\Biggr).\end{aligned} \tag{III.22}$$

We remark that the transformation formula is obtained if the second part of the sum in the latter expression vanishes. To this end, we rewrite this part as a sum over odd integers:

$$\sum_{l\in\mathbb{Z}} \frac{1}{\sqrt{3}} \exp\left[\pi i \left(\frac{\tau(6l+3)^2}{12} - \frac{(6l+3)}{6}\right)\right] = \sum_{m\in 2\mathbb{Z}+1} \frac{1}{\sqrt{3}} \exp\left[\pi i \left(\frac{3\tau m^2}{4} - \frac{m}{2}\right)\right] . \tag{III.23}$$

This last expression can be separated into a sum over positive and negative integers:

$$\sum_{\substack{m\in 2\mathbb{Z}+1 \\ m>0}} \frac{1}{\sqrt{3}} \exp\left[\pi i \left(\frac{3\tau m^2}{4}\right)\right] \left(e^{\pi i m/2} + e^{-\pi i m/2}\right) . \tag{III.24}$$

However, $e^{\pi i m/2} + e^{-\pi i m/2} = 2\cos\left(\frac{m\pi}{2}\right)$ and this function attains zero at all $m \in 2\mathbb{Z}+1$. Therefore the entire sum in (III.23) vanishes, so that the second transformation formula is found by combining (III.20) with (III.22). □

The transformations covered in the above theorem, namely $\tau \to \tau+1$ and $\tau \to \frac{-1}{\tau}$, are particular elements of a much broader class of transformations, the *Möbius transformations*. The theory of Möbius transformations is discussed in the following paragraph, together with some principal concepts concerning their nature.

III C. Möbius transformations of the complex upper half-plane

A key concept in the building of Möbius transformations is $\mathrm{SL}_2(\mathbb{Z})$, the group of (2×2)-matrices with integer entries and determinant equal to unity. The fact that $\mathrm{SL}_2(\mathbb{Z})$ indeed forms a group under matrix multiplication, is proven in the following proposition:

Lemma III.1. *Define* $SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a,b,c,d \in \mathbb{Z},\ \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1 \right\}$. *Then* $SL_2(\mathbb{Z})$ *is a group for the operation of matrix multiplication.*

Proof. Since $\det(AB) = \det(A)\cdot\det(B)$ is true for any pair of $(n\times n)$-matrices A and B, the product matrix of two matrices having determinant equal to unity has determinant equal to unity as well. Together with the fact that the product of two matrices with integer entries is again a matrix with integer entries, we find that $\mathrm{SL}_2(\mathbb{Z})$ is closed under matrix multiplication. Seen as this operation is associative for all matrices, it is associative for $\mathrm{SL}_2(\mathbb{Z})$ in particular.

The identity element of $\mathrm{SL}_2(\mathbb{Z})$ is the identity matrix $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.

The inverse element of a (2×2)-matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

for matrix multiplication is given by

$$A^{-1} = \det(A)^{-1} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} .$$

If $\det(A) = ad - bc = 1$, then A^{-1} has determinant equal to unity as well. It follows that if $A \in \mathrm{SL}_2(\mathbb{Z})$, then $A^{-1} \in \mathrm{SL}_2(\mathbb{Z})$. □

The above result allows us to present a precise definition for the Möbius transformations through the following group action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$:

$$\mathrm{SL}_2(\mathbb{Z}) \times \mathbb{H} \to \mathbb{H} : \left(\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tau\right) \mapsto \gamma \cdot \tau = \frac{a\tau + b}{c\tau + d} . \tag{III.25}$$

In other words, the group $\mathrm{SL}_2(\mathbb{Z})$ acts on the complex upper half-plane via Möbius transformations. The dot is often omitted so that $\gamma\tau$ is written for $\gamma \cdot \tau$. We have yet to verify that this action is well defined by showing $\mathbb{H}$ is stable under the action of $\mathrm{SL}_2(\mathbb{Z})$, i.e. $\gamma\tau \in \mathbb{H}$ for every $\tau \in \mathbb{H}$. This is the subject of the following lemma:

Lemma III.2. *If* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ *and* $\tau \in \mathbb{H}$, *then*

$$\mathrm{Im}(\gamma\tau) = \frac{\mathrm{Im}(\tau)}{|c\tau + d|^2} . \tag{III.26}$$

Proof. Direct computation of the imaginary part of $\gamma\tau$ gives:

$$\begin{aligned} \mathrm{Im}(\gamma\tau) &= \mathrm{Im}\left(\frac{a\tau + b}{c\tau + d}\right) \\ &= \frac{1}{|c\tau + d|^2}(ad - bc)\mathrm{Im}(\tau) . \end{aligned}$$

Since $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and so $ad - bc = 1$, we find the desired result. □

This lemma implies that if $\mathrm{Im}(\tau) > 0$, as is the case for $\tau \in \mathbb{H}$, then $\mathrm{Im}(\gamma\tau) > 0$, so that also $\gamma\tau \in \mathbb{H}$. The action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$ is as such well defined.

Two elements of $\mathrm{SL}_2(\mathbb{Z})$ are of particular interest:

$$\begin{aligned} T &:= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\ S &:= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} . \end{aligned} \tag{III.27}$$

These transformations are referred to as the *translation* and *(modular) inversion* respectively. It can easily be shown that

$$T^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} . \tag{III.28}$$

The reason why these particular transformations are of great interest is because they generate $\mathrm{SL}_2(\mathbb{Z})$, as is the subject of the following theorem. The proof will not be stated here, but can be found in [9].

Theorem III.2. *The matrices T and S are the generators of $SL_2(\mathbb{Z})$. This means that any matrix γ from $SL_2(\mathbb{Z})$ can be written as a combination of products of these two matrices.*

The fact that S and T are the generators of $\mathrm{SL}_2(\mathbb{Z})$ has profound implications on the Dedekind function η. After all, a natural question to ask is how the Dedekind function behaves under a general Möbius transformation of the complex upper half-plane. This topic is elaborated on in the following paragraph.

III D. A central result concerning Möbius transformations and η

We now motivate our study of the general theory of Möbius transformations by explicating the effect of such a transformation to η. As will become clear, η exhibits some interesting symmetries with regard to these transformations.

Using the notation introduced in (III.27), the transformation formulae (III.10) of η become

$$\begin{aligned} \eta(T\tau) &= e^{\frac{i\pi}{12}}\eta(\tau) \\ \eta(S\tau) &= (-i\tau)^{\frac{1}{2}}\eta(\tau) \end{aligned} \tag{III.29}$$

If we let $c_S = 1$ and $d_S = 0$, respectively $c_T = 0$ and $d_T = 1$, denote the c- and d-entries of S, respectively T, then we can derive the following equalities that are needed to reformulate (III.29):

$$\begin{aligned} (-i\tau)^{\frac{1}{2}} &= e^{\frac{3i\pi}{4}}(c_S\tau + d_S)^{\frac{1}{2}} \\ 1 &= (c_T\tau + d_T)^{\frac{1}{2}} \, . \end{aligned} \tag{III.30}$$

Combining (III.29) with the newly obtained equalities, we get that

$$\begin{aligned} \eta(T\tau) &= e^{\frac{i\pi}{12}}(c\tau + d)^{\frac{1}{2}}\eta(\tau) \\ \eta(S\tau) &= e^{\frac{3i\pi}{4}}(c\tau + d)^{\frac{1}{2}}\eta(\tau) \, , \end{aligned} \tag{III.31}$$

where we have dropped the subscripts of c and d as it is clear they refer to the entries of the transformation of τ on the left hand side. If we ignore the phase factors in front of the expressions on the right hand side, we see that η exhibits a special symmetry with respect to T and S. This symmetry becomes exact if we get rid of those phase factors. The way to do this is by raising the obtained expressions to a power of 24, as this is the smallest number making each of the phase factors equal to one. The result of this operation is

$$\begin{aligned} \eta^{24}(T\tau) &= (c\tau + d)^{12}\eta^{24}(\tau) \\ \eta^{24}(S\tau) &= (c\tau + d)^{12}\eta^{24}(\tau) \, . \end{aligned} \tag{III.32}$$

We now invoke the result of the preceding subsection that S and T are generators of $\mathrm{SL}_2(\mathbb{Z})$. The fact that any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ can be written as a string of products of S and T means that the symmetry exhibited by η^{24} with respect to S and T can be generalized to a symmetry of η^{24} with respect to any Möbius transformation γ. We obtain the central result of this section:

$$\eta^{24}\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{12}\eta^{24}(\tau) \, , \tag{III.33}$$

for every $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. The function η^{24} is often denoted by Δ and is called the modular discriminant function.

The fact that Δ displays these symmetries is of paramount importance for the continuation of our study into the behaviour of $\mathcal{Z}$, as these symmetries have been widely studied in the context of a more general group of functions with the same property as Δ and that bear the name of modular forms. The rest of this paper is now concerned with the study of properties of modular forms and how they translate to features of Δ, and as such to features of $\mathcal{Z}$.

IV The general theory of modular forms

In the last section we concluded that the modular discriminant function exhibits symmetry properties under the action of Möbius transformations. We now study these properties as particular examples of modular symmetry by introducing the general definition of modular forms, followed by a discussion of a few important results concerning these highly symmetric functions. We bring forward their periodic character and hereby study their Fourier series. The Fourier coefficients of the partition function $\mathcal{Z}$ we have been considering up until now and for which we have established a connection to modularity, will turn out to contain information about the physics of the system. In the light of the interest in the limiting behaviour of our system, it is also worthwile to investigate the growth of the Fourier coefficients.

IV A. Basic definitions and examples

We introduce a definition of modular forms, which is directly related to our foray at the end of the previous section into the general transformation formula for $\Delta = \eta^{24}$:

Definition IV.1. A function $f : \mathbb{H} \to \mathbb{C}$ is called a modular form of weight $k \in \mathbb{Z}$ for $\mathrm{SL}_2(\mathbb{Z})$ if and only if f satisfies the functional equation:

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau)\,, \tag{IV.1}$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.

Upon recalling the transformation formula (III.33) for the modular discriminant function, we conclude from the above definition that Δ is a modular form of weight 12.

The purpose of our study of modular forms is to obtain results from which we are able to deduce implications to the physics of our system. With this goal in mind, the study of modular forms appears to be a futile enterprise in the investigation of $\mathcal{Z}$, as $\mathcal{Z}$ is related to η and η itself does not conform to the definition of a modular form for two reasons: its weight would be $\frac{1}{2}$, which is not integral, and it allows for extra phase factors, as can be seen in (III.31).

However, η satisfies a more general definition by virtue of which it can be of half integral weight and can allow for phase factors. The theorems that are relevant to our discussion of $\mathcal{Z}$ will still hold for these more general functions, as we will argue when we introduce them. We nonetheless still adopt definition IV.1 if we wish to talk about modular forms and their properties. The more general definition for modular forms with the purpose of including η becomes:

Definition IV.2. A function $f : \mathbb{H} \to \mathbb{C}$ is called a modular form of weight $k \in \mathbb{Z} + \frac{1}{2}$ or $k \in \mathbb{Z}$ with multiplier system ε for $\mathrm{SL}_2(\mathbb{Z})$ if and only if f satisfies the functional equation

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = \varepsilon(a, b, c, d)(c\tau + d)^k f(\tau)\,,$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and for which $|\varepsilon(a, b, c, d)| = 1$. If for all $a, b, c, d \in \mathbb{Z}$, $\varepsilon(a, b, c, d) = 1$ holds, then the multiplier system is said to be trivial. In this case, f is just called a modular form of weight k.

With this definition in mind, we can now say that η is a modular form of weight $\frac{1}{2}$ with a certain multiplier system ε yet to be explicated. Because of the result that $\mathcal{Z}$ is the reciprocal of the Dedekind eta function η, it is easy to see from the above definition that $\mathcal{Z}$ is also a modular form, of weight $-\frac{1}{2}$, along with a certain multiplier system ε.

In this paper we will not delve deeper into the derivation of the precise form of the multiplier systems ε. An important remark however is that the multiplier system does not depend on the argument of the modular form, τ, but only on the parameters of the transformation γ, i.e. a, b, c and d, and on the function f itself. Furthermore, we remark that multiplier systems, albeit introduced here in the context of modular forms, can be studied separately. See for instance [8].

An immediate consequence of definition IV.1 is that there are no non-zero modular forms of odd weight. Indeed, if f is a modular form of weight $k = 2m + 1$ for $m \in \mathbb{Z}$ and we take $\gamma = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, then the modularity of f implies that for any $\tau \in \mathbb{H}$:

$$f(\gamma\tau) = f(\tau) = (-1)^{2m+1} f(\tau) = -f(\tau)\,, \tag{IV.2}$$

from which we deduce that $f(\tau) = 0$ for all $\tau \in \mathbb{H}$.

Another important consequence of the definition of modular forms is the fact that modular forms admit periodicity. Indeed, let $\gamma = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and f be a modular form of weight k. Then $f(\gamma\tau) = f(\tau + 1) = f(\tau)$. This implies that modular forms possess a Fourier series:

$$f(\tau) = \sum_{n\in\mathbb{Z}} \alpha_n e^{2\pi i n\tau} = \sum_{n\in\mathbb{Z}} \alpha_n q^n\,. \tag{IV.3}$$

The Fourier expansion of a modular form is also frequently called the q-series. We will study properties of the Fourier coefficients of modular forms in more detail in the next section.

IV B. The vector space of modular forms of weight k

In this subsection, we will show that the modular symmetry is preserved under the additive operation of two modular forms of weight k and under scalar multiplications of modular forms with complex numbers. This then allows us to study vector spaces of modular forms of weight k over the field $\mathbb{C}$. Moreover, the structure of the Fourier expansion of modular forms naturally gives rise to a subdivision of these spaces based on the smallest power of q for which the Fourier coefficient is not zero.

We now make the argument concerning vector spaces of modular forms of weight k precise. Let f and g be two modular forms of weight $k \in \mathbb{Z}$. A simple calculation shows that the sum $f + g$ is again a modular form of weight k. After all, for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $\tau \in \mathbb{H}$:

$$(f + g)(\gamma\tau) = f(\gamma\tau) + g(\gamma\tau) = (c\tau + d)^k (f + g)(\tau)\,. \tag{IV.4}$$

If f is a modular form of weight k and $\lambda \in \mathbb{C}$ is a complex number, then λf is again modular form of weight k, as can readily be seen. In summary, modular forms of weight k form a vector space over $\mathbb{C}$. We denote this vector space by $\mathcal{M}_k$.

Recall from equation (IV.3) that modular forms admit a Fourier expansion. We now consider a subdivision in the space of modular forms of weight k based on the nature of their Fourier coefficients. If the Fourier coefficients α_n of a modular form f are zero for $n \leq 0$, then f is said to be a cusp form. It is easily verified that if two modular forms f and g of weight k satisfy this property, then their sum $\lambda f + \mu g$ ($\lambda, \mu \in \mathbb{C}$) satisfies it as well. Therefore, cusp forms of weight k comprise a subspace of modular forms of weight k. We denote this subspace by $\mathcal{M}_k^0$. If it is true that $\alpha_n = 0$ for $n < 0$, but $\alpha_0 \neq 0$, then f is said to be a holomorphic modular form of weight k. The nomenclature "holomorphic" stems from complex analytic considerations of this function. If for $n < 0$ there possibly is a Fourier coefficient α_n of f that is non-zero, and the sum in the Fourier expansion is not over the whole set of integers $\mathbb{Z}$, then f is said to be a weakly holomorphic modular form of weight k. Again, weakly holomorphic modular forms constitute a vector space over $\mathbb{C}$, which we denote by $\mathcal{M}_k^!$. From these definitions it is quite straightforward to see that the following inclusions hold:

$$\mathcal{M}_k^0 \subset \mathcal{M}_k \subset \mathcal{M}_k^! \,. \tag{IV.5}$$

In the light of future assertions that will be made about the dimensions of these vector spaces, we add the observation that the product of a modular form f of weight k and a modular form g of weight l is again a modular form, but now of weight $k + l$. The product of a modular form f of weight k and a cusp form g of weight l is a cusp form of weight $k + l$.

We illustrate the above by displaying the series representation of the modular discriminant function $\Delta = \eta^{24}$. It can be obtained with the help of the pentagonal number theorem. Recall that

$$\Delta(q) = q \prod_{k=1}^{+\infty} (1 - q^k)^{24} \,. \tag{IV.6}$$

The pentagonal number theorem then shows that

$$\prod_{k=1}^{+\infty} (1 - x^k) = \sum_{k=-\infty}^{+\infty} (-1)^k x^{(3k^2 - k)/2} \,, \tag{IV.7}$$

i.e. there are no negative powers of q appearing in the series expansion of Δ. Moreover, the right hand side of (IV.7) has a constant term equal to 1. However, after multiplication with q in front of this series expansion, we find that Δ has no non-zero constant term in its q-series. Therefore, we can conclude that Δ is a cusp form of weight 12, i.e. $\Delta \in \mathcal{M}_{12}^0$.

Recall that there are no non-zero modular forms of odd weight. This essentially means that $\mathcal{M}_{2k+1} = \{0\}$ for all $k \in \mathbb{Z}$. It is a natural question to ask whether the vector space of modular forms of weight k with k even is finite dimensional, and if so, whether a closed form expression for the calculation of the dimension exists. A quite remarkable result shows that the answer to both of these questions is positive. This is precisely the content of the following theorem, of which the proof can be found in [10].

Theorem IV.1. *If $k < 0$ or if k is odd, then $\dim_{\mathbb{C}} \mathcal{M}_k = 0$. For $k \geq 0$ and k even, we have*

$$\dim_{\mathbb{C}} \mathcal{M}_k = \begin{cases} \lfloor k/12 \rfloor + 1 & \text{if } k \not\equiv 2 \ mod\ 12 \\ \lfloor k/12 \rfloor & \text{if } k \equiv 2 \ mod\ 12 \end{cases} ,$$

where $\lfloor a \rfloor$ denotes the integer part of $a \in \mathbb{R}$.

An immediate consequence of the above theorem is that there are no non-zero modular forms of weight 2, since $\dim_{\mathbb{C}} \mathcal{M}_2 = \lfloor 2/12 \rfloor = 0$. The dimensions of the vector spaces $\mathcal{M}_4$ and $\mathcal{M}_6$ are both equal to one. This means that, up to a complex prefactor, there exists only one modular form of weight 4 and weight 6 respectively. These modular forms are called the Eisenstein series of weight 4 and weight 6, which we specify below.

The Eisenstein series of weight $k \in \mathbb{Z}$ for $k \geq 3$ is defined as the function:

$$G_k : \mathbb{H} \to \mathbb{C} : \tau \mapsto G_k(\tau) = \sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} \frac{1}{(m\tau+n)^k} \,. \tag{IV.8}$$

The Eisenstein series of weight k is a modular form of weight k. Indeed, take an arbitrary $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and any $\tau \in \mathbb{H}$. We calculate:

$$\begin{aligned} G_k(\gamma\tau) &= \sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} \left(m\frac{a\tau+b}{c\tau+d}+n\right)^{-k} \\ &= (c\tau+d)^k \sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} ((ma+nc)\tau+mb+nd)^{-k} \,. \end{aligned} \tag{IV.9}$$

But if (m,n) runs through all of $\mathbb{Z}^2 \setminus (0,0)$, then also $(ma+nc, mb+nd)$ runs through all of $\mathbb{Z}^2 \setminus (0,0)$ as well. Therefore we get

$$G_k(\gamma\tau) = (c\tau+d)^k G_k(\tau) \,, \tag{IV.10}$$

so that the Eisenstein series of weight k is a modular form of weight k. As we already stated, there are no non-zero modular forms of odd weight, so this most certainly implies that the Eisenstein series of odd weight vanishes. However, it is a priori not clear if the Eisenstein series for $k \geq 4$ and k even do *not* vanish, since this would make the definition in (IV.8) meaningless. Luckily, a proof of the non-vanishing of the Eisenstein series for $k \geq 4$ and k even is found in [7].

By virtue of theorem IV.1, G_4 and G_6 are the only modular forms of weight 4 and weight 6 respectively, albeit up to a complex factor. We are allowed to normalise these Eisenstein series in the sense that the constant term in the series expansion of G_4 and G_6 becomes 1 through multiplication with a proper complex prefactor. The normalised Eisenstein series of weight k is denoted by E_k. The first few terms of the series expansions of E_4 and E_6 are given by (see [7])

$$\begin{aligned} E_4(\tau) &= 1 + 240q + 2160q^2 + \dots \\ E_6(\tau) &= 1 - 504q - 16632q^2 + \dots \,. \end{aligned} \tag{IV.11}$$

Note that both Eisenstein series are holomorphic modular forms since both series expansions start off with a non-zero constant term and do not consist of negative powers of q.

As we have already demonstrated before, the multiplication of modular forms results in a new modular form of weight equal to the sum of the weights of the original modular forms. Therefore, we find that E_4^2 is a modular form of weight 8. As a result of theorem IV.1, this must be the only modular form of weight 8, up to a complex factor. The same reasoning shows that E_4E_6 is the only modular form of weight 10, up to a complex factor. The dimension of $\mathcal{M}_{12}$ is equal to two, and since E_6^2 and E_4^3 constitute two linearly independent modular forms of weight 12, as can be seen from the first few coefficients of their series expansion, any modular form f of weight 12 can be expanded as:

$$f = \alpha E_4^3 + \beta E_6^2 \,. \tag{IV.12}$$

As a particular example of this last assertion, we recall that the modular discriminant function, $\Delta(\tau) = \eta^{24}(\tau)$, is a modular form of weight 12. Hence we are able to write Δ in terms of the Eisenstein series E_4 and E_6 as displayed in (IV.12). Since we have already proven that Δ is a cusp form and therefore has no constant term in its q-series, and since both E_4 and E_6 do in fact have a constant term of 1 in their series expansion, we from this deduce that the sum has to be of the form $\Delta = \alpha(E_4^3 - E_6^2)$ in order to get rid of the constant term. By equating the series expansion of Δ to this combination of E_4 and E_6, one finds that (see [7])

$$\Delta = \frac{E_4^3 - E_6^2}{1728}\,. \tag{IV.13}$$

This principle can be extended to any vector space of modular forms of a specific weight k. The general statement of this fact is given in the following theorem, of which Jean-Pierre Serre provides a proof in [10]:

Theorem IV.2. *For $k \geq 4$ even, the set*

$$\{E_4^a E_6^b \mid a, b \in \mathbb{N}, 4a + 6b = k\} \tag{IV.14}$$

forms a basis for the finite dimensional vector space $\mathcal{M}_k$.

This precisely means that for any modular form f of weight k, the first few coefficients of the q-series of f contain sufficient information to determine the coefficients in the linear combination of the basis functions $E_4^a E_6^b$ that produces f.

IV C. Asymptotic growth of Fourier coefficients of modular forms

The previous paragraph demonstrated that the Fourier coefficients of modular forms bring about a natural distinction of spaces of modular forms into cusp, holomorphic and weakly holomorphic modular forms respectively. However, the implications that are established by these Fourier coefficients of modular forms to mathematics and physics do not end there. There exist some well-known open problems in the mathematical field of number theory concerning integer sequences that comprise the Fourier coefficients of certain modular forms. As an introduction to this matter, we start off with a brief discussion of these specific mathematical problems. Moreover, we introduce a general theorem about the asymptotic growth of the considered Fourier coefficients. The analysis of the Fourier coefficients of $\mathcal{Z}$ will prove vital in the gathering of information concerning the physical aspects of our system.

The Ramanujan τ-function

The Fourier coefficients of the modular discriminant function Δ are often denoted by $\tau(n)$. The function $\tau : \mathbb{N} \to \mathbb{Z} : n \mapsto \tau(n)$ is called the Ramanujan tau function. Long before it was recognised that Δ exhibited modular properties, Ramanujan and contemporaries already investigated into the nature of the integer sequence $\tau(n)$. Many properties of the tau function were first brought about as conjectures from the hand of Ramanujan, such as the fact that τ is a multiplicative function, which was later proven by Mordell using techniques from the theory of modular forms; see for instance [6]. All of these conjectures were centered around the Ramanujan hypothesis, which was formulated in 1916 by Ramanujan and which states that

$$|\tau(p)| < 2p^{11/2}\,, \tag{IV.15}$$

for p prime. A rigorous proof of this statement was given almost 60 years later by Pierre Deligne. Derrick Henry Lehmer conjectured that $\tau(n) \neq 0$ for all n, which was at that time known as Lehmer's conjecture of the non-vanishing of the Ramanujan tau function. The conjecture was proven by Will Y. Lee in 2015, see [5].

The partition of integers

We recall that the partition function for the system of an infinite amount of quantum harmonic oscillators is given by

$$\mathcal{Z}(q) = \eta^{-1}(q) = q^{-\frac{1}{24}} \prod_{k=1}^{+\infty} \frac{1}{1-q^k} \,. \tag{IV.16}$$

The infinite product appearing in the left hand side of this expression can be written in a series expansion:

$$\prod_{k=1}^{+\infty} \frac{1}{1-q^k} = \sum_{k=1}^{+\infty} p(n) q^n \,. \tag{IV.17}$$

We are now able to study the coefficients $p(n)$ that appear in the series expansion through knowledge of the product on the left hand side and vice versa. This method of encoding a sequence as the coefficients of a series expansion is called the method of the generating function. It is an extensively studied function in number theory, not in the least because of the following non-trivial result, which we will not prove here: the coefficients $p(n)$ represent the number of ways to partition an integer n. A partition of a positive integer n is a way of writing n as a sum of smaller positive integers. $p(n)$ then counts the number of ways n can be written as such a sum, for which the order of terms in the summation does not matter and for which zero is not included in the sum. For example: $3 = 2 + 1$ and $3 = 1 + 1 + 1$, so $p(3) = 2$. Hardy and Ramanujan explicated the asymptotic behaviour of $p(n)$, which is the content of the so-called Hardy-Ramanujan asymptotic formula:

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi \sqrt{\frac{2}{3}} \sqrt{n}} \,, \tag{IV.18}$$

for $n \to +\infty$. The take-away message concerning this result is the fact that $p(n)$ grows exponentially, in the form $e^{c\sqrt{n}}$ with $c = \pi\sqrt{\frac{2}{3}}$. The growth of such an exponential is very sensitive to the specific values of c. To get a feeling for this sensitivity, a plot of the first 50 values of $p(n)$ along with the asymptotic formula of Hardy-Ramanujan and two other estimates of c in the same formula is provided in figure 1. The Hardy-Ramanujan estimate uses the value of $c_{HR} := \pi\sqrt{\frac{2}{3}} \approx 2,5651$. The asymptotic formula that incorporates the other two estimates of c exhibits the same exponential behaviour as in (IV.18), but for the values $c = c_{HR} - 0,1$ and $c = c_{HR} + 0,1$ respectively. Even for the partition of the first 50 integers and for such a small difference with the estimate of c obtained by Hardy and Ramanujan, the discrepancy in the growth is clearly visible.

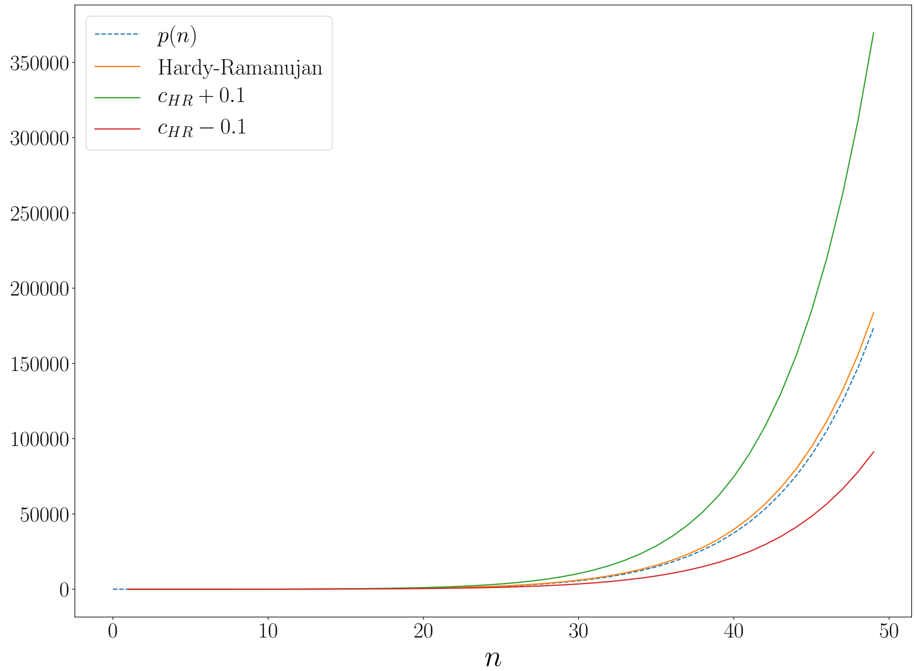

Figure 1: The Hardy-Ramanujan asymptotic formula tested for the first 50 values of $p(n)$. The values of $p(n)$ were obtained from the Online Encyclopedia of Integer Sequences, sequence A000041.

Implementing the expansion in (IV.17) into (IV.16) yields

$$\mathcal{Z}(q) = q^{-\frac{1}{24}} \sum_{n=1}^{+\infty} p(n) q^n \,. \tag{IV.19}$$

We therefore conclude that a study of the partition of integers is of great interest to us if we wish to understand asymptotical properties of $\mathcal{Z}$.

A central result concerning asymptotics of Fourier coefficients of modular forms

It turns out that the Fourier coefficients of the three different kinds of modular forms, i.e. cusp, holomorphic and weakly holomorphic, possess different growth properties for large n. This is the essence of the following theorem, also valid for modular forms with multiplier system ε and which is proven in [2]:

Theorem IV.3. *Let f be a modular form of weight k and let α_n denote the n-th Fourier coefficient of f. We get:*

1. *If $f \in \mathcal{M}_k^0$, then $\alpha(n) \sim \mathcal{O}(n^{k/2})$ for $n \to \infty$*
2. *If $f \in \mathcal{M}_k$, then $\alpha(n) \sim \mathcal{O}(n^{k-1})$ for $n \to \infty$*
3. *If $f \in \mathcal{M}_k^!$, then $\alpha(n) \sim \mathcal{O}(e^{c\sqrt{n}})$ for $n \to \infty$, for which $c > 0$.*

Note that the asymptoticality result of the partition of integers is in agreement with this theorem. Since Δ is a cusp form of weight 12, we expect that its Fourier coefficients will be limited by a polynomial growth of the form $\mathcal{O}(n^6)$ for n going off to infinity, which is in accord with the result by Ramanujan given in (IV.15). Moreover, Ramanujan's hypothesis does an even better job at estimating the asymptotic growth than theorem IV.3.

V The physics behind the Fourier coefficients of $\mathcal{Z}$

In the previous section we introduced the theory of modular forms and discussed results about their Fourier coefficients. We now relate these findings to physical results on our system of an infinite amount of quantum harmonic oscillators.

V A. The number of eigenstates corresponding to the energy eigenvalues

We recall that the partition function of a discrete quantum mechanical system is given by the definition

$$\mathcal{Z}(\beta) = \sum_{n\in\mathbb{N}} d(n) e^{-\beta E_n} \,, \tag{V.1}$$

where $d(n)$, the degeneracy factor, denotes the number of eigenstates corresponding to an energy E_n. In the particular case of a system of an infinite amount of quantum harmonic oscillators, we found that $\mathcal{Z} = \eta^{-1}$. As this is a modular form, the preceding reflections about its Fourier series representation allow us to relate these Fourier coefficients to the number of states corresponding to a certain energy value from the discrete energy spectrum. To this end we recall that we performed changes of variables from β to τ, and from τ to q. Implementing these changes of variables in (V.1) and upon comparison with the obtained q-series of $\mathcal{Z}$, it follows that $d(n) = \alpha_n$. In essence, the number of eigenstates corresponding to energy E_n is determined by the Fourier coefficient α_n.

V B. The entropy and its asymptotic growth

Seen as we were able to find an identification of the Fourier coefficient α_n with the number of eigenstates corresponding to E_n, the Boltzmann relation then tells us that the entropy of this system corresponding to energy E_n is given by $S_n = k_b \log \alpha_n$. We are particularly interested in the growth of the entropy as we increase the energy of the system, since we have derived vital asymptoticality results for α_n. As pointed out earlier, the partition function $\mathcal{Z}$ constitutes a weakly holomorphic modular form and therefore it follows from theorem IV.3 that the Fourier coefficients α_n experience exponential growth as n goes off to infinity. We can even get more precise than that: we have already established the result that the sequence of Fourier coefficients α_n is identical to the sequence of partition of integers $p(n)$, as was pointed out in (IV.19). Therefore, the Hardy-Ramanujan asymptotic formula gives an explicit expression for this exponential growth of the α_n, and thus yields the explicit polynomial growth of the entropy S:

$$S \sim \pi\sqrt{\frac{2}{3}}\sqrt{n} \,, \tag{V.2}$$

as n goes to infinity. This is a fundamental result that will allow for a transition to conformal field theory, as we set out to motivate in the outlook of this project.

VI Outlook & Conclusion

VI A. Outlook

The general theory of modular forms covered in this paper along with the results we have derived for $\mathcal{Z}$, make frequent appearances in various fields of physics. In this paragraph, we briefly discuss a few of these applications without delving too deeply into the general theory behind it, as these topics merely aspire to be a way of displaying the importance of modular symmetry in several branches of theoretical and mathematical physics.

Quantum field theory

It is a natural question to ask why we are interested in this seemingly pathological system of an *infinite* amount of quantum harmonic oscillator, as for the study of regular quantum mechanics this consideration yields no particularly useful results. An explanation can be found in the claim that this system serves as a stepping stone to the more general theory of quantum field theory (QFT).

Quantum field theory no longer deals with classical point particles, but rather describes the physical reality as being comprised of a field continuum. Useful results concerning the physics of this continuum can be extracted using raising and lowering operators similar to the creation and annihilation of the quantum harmonic oscillator. As such, a field continuum necessitates a description of the behaviour of oscillators in the limiting case for which we consider an infinite amount of them.

Conformal field theory

A pivotal concept in conformal field theory (CFT) is the study of operators that act on the complex plane $z = x + iy \in \mathbb{C}$ and that are of the form

$$L_n = -z^{n+1}\frac{\partial}{\partial z} \tag{VI.1}$$

for which n is an integer. It is then possible to show that all the symmetrical transformations on the complex upper half-plane $\mathbb{H}$ can be constructed from arithmetical combinations of these operators. For instance, the operator $-\frac{1}{2}(L_{-1} + \bar{L}_{-1} = \frac{\partial}{\partial x})$ represents infinitesimal translations in the x-direction of the complex plane. Through introduction of the commutation relation

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}\,, \tag{VI.2}$$

for which δ represents the Kronecker delta, a fundamental algebra can be obtained, named after Virasoro. Without providing further details we mention that the Virasoro algebra is also called the algebra of symmetries of a 2D CFT, as it contains translations, rotations, dilatations and the alleged special conformal boosts. The quantity c in (VI.2) is the central extension or central charge of the CFT and can be regarded as a way of characterising the particular CFT that is considered.

It was shown in 1986 by John Cardy that the entropy of a two dimensional CFT is given by the Cardy formula

$$S = 2\pi\sqrt{\frac{c}{6}\left(L_0 - \frac{c}{24}\right)}\,, \tag{VI.3}$$

where c again represents the central charge of the system, and L_0 can be thought of as a generalisation of the energy. It can be proven that a 2D CFT with central charge equal to one provides the description of the excited states of a boson moving freely on a 2-torus. If we substitute this value for c into (VI.3), we find that the entropy described by the Cardy formula is identical to the entropy of the system of an infinite amount of quantum harmonic oscillators. In a sense, this suggests that the system of oscillators we have been considering throughout this paper constitutes an equivalent description of free bosonic movement on a 2-torus.
A possible continuation of the theory in this paper could be the investigation of the transition from quantum mechanics to conformal field theory. It is certainly possible that other systems

in CFT can be described by partition functions that satisfy modular symmetries analogous to $\mathcal{Z}$, as we have motivated here for the case of the bosonic particle on a 2-torus.

Possible generalisations of modular forms

There are a few possible continuations to the theory we have presented that go deeper into the mathematics behind modular forms and that generalise this concept. Without providing further details, some of these generalisations include mock modular forms, theta functions and Jacobi forms.

As was the case for modular forms, these generalisations appear throughout the whole of theoretical physics. For example, the quantum degeneracies of single-centered black holes can be connected to the Fourier coefficients of mock Jacobi forms, as is covered in [2]. This complements the discussion in this paper, where we concluded that the degeneracy factors of the energy eigenvalues of the infinite amount of quantum harmonic oscillators are exactly the Fourier coefficients of the weakly holomorphic modular form $\frac{1}{\eta}$.

VI B. Conclusion

We demonstrated that the partition function of an infinite amount of quantum harmonic oscillators is given by the reciprocal of the Dedekind eta function. We observed that the Dedekind eta function satisfies highly symmetrical properties, which then led us to the more general study of modular forms. A reflection on some fundamental and deep results about vector spaces of modular forms and their Fourier coefficients followed, in order to extract meaningful physical information about the system we were considering. The central result of this paper is that the Fourier coefficients of the partition function $\mathcal{Z}$, obtained through the foray into modular forms, is identical to the degeneracy factor or number of states corresponding to a certain eigenvalue of the energy spectrum. Upon application of the Boltzmann relation, this then led to the conclusion that the entropy of the system satisfies the asymptotic behaviour as described by (V.2). We motivated that the study of the concepts does not end on this note: a suggestion for possible continuations of the theory concerning the physical system and generalisations of modular forms has been laid bare.

Regarding the aspiration to bring the interesting interplay between mathematics and physics into light, we wish to add that we were highly dependent on results stemming from number theory, complex analysis and algebra so as to be able to obtain physical information about our original system. This confirms the assertion made in the introduction that to fully comprehend nature in its most fundamental appearances, we need a rigorous mathematical framework to work in.

Appendices

A Solution of the quantum harmonic oscillator by the ladder operator method

Consider a 1D quantum harmonic oscillator given by the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 \tag{A.1}$$

We introduce new variables, $X = \sqrt{\frac{m\omega}{2\hbar}}\, x$ and $P = \sqrt{\frac{1}{2m\omega\hbar}}\, p$. Expressed in these variables, the Hamiltonian is:

$$H = \hbar\omega(P^2 + X^2) \tag{A.2}$$

Note that $[X, P] = \frac{1}{2}i$. The creation and annihilation operators, also called the ladder operators, are defined as:

$$\begin{aligned} a &= \frac{1}{\sqrt{2}}\left[\left(\frac{m\omega}{\hbar}\right)^{1/2} x + i\frac{p}{(m\hbar\omega)^{1/2}}\right] \\ a^\dagger &= \frac{1}{\sqrt{2}}\left[\left(\frac{m\omega}{\hbar}\right)^{1/2} x - i\frac{p}{(m\hbar\omega)^{1/2}}\right] \end{aligned} \tag{A.3}$$

And in terms of the new variables X and P, the ladder operators are:

$$a = X + iP, \quad a^\dagger = X - iP \tag{A.4}$$

The ladder operators satisfy the commutation relation $[a, a^\dagger] = \mathbb{1}$. In order to show this, we apply the definition of the ladder operators from (A.4).

$$\begin{aligned} \left[a, a^\dagger\right] &= aa^\dagger - a^\dagger a \\ &= -2i[X, P] = \mathbb{1} \end{aligned}$$

The Hamiltonian can be expressed in yet another form, by introducing a new operator, called the number operator, $N = a^\dagger a$. Expanded, this operator can be written as

$$N = a^\dagger a = X^2 + P^2 - \frac{1}{2}\mathbb{1}$$

Therefore, $N + \frac{1}{2}\mathbb{1} = X^2 + P^2$, and by using (A.2), the Hamiltonian can be written as:

$$H = \hbar\omega\left(N + \frac{1}{2}\right) \tag{A.5}$$

The eigenvalues of the Hamiltonian are denoted by $|E\rangle$. We claim that whenever $H|E\rangle = E|E\rangle$, then $Ha^\dagger|E\rangle = (E + \hbar\omega)a^\dagger|E\rangle$ and a similar relation holds when $a^\dagger$ is replaced by a, and the energy value gets lowered by an amount $\hbar\omega$. This means that if $|E\rangle$ is an eigenstate of the Hamiltonian, then $a|E\rangle$ and $a^\dagger|E\rangle$ are also eigenstates of the Hamiltonian. Note that $a^\dagger$ creates an eigenstate corresponding to a higher energy, while a creates an eigenstate corresponding to a lower energy. In both cases, the difference between adjacent energy levels is $\hbar\omega$.

To show this, note that by manipulating the commutator of H with one of the ladder operators using standard properties of commutators gives the following results: $[H, a] = -\hbar\omega a$ and $[H, a^\dagger] = \hbar\omega a^\dagger$. Now note that $Ha = [H, a] + aH$ and similarly $Ha^\dagger = [H, a^\dagger] + a^\dagger H$. Now we let these operator act on an eigenstate $|E\rangle$:

$$\begin{aligned} Ha^\dagger|E\rangle &= \left([H, a^\dagger] + a^\dagger H\right)|E\rangle \\ &= (E + \hbar\omega)\, a^\dagger|E\rangle \end{aligned}$$

And similarly one finds that $Ha|E\rangle = (E - \hbar\omega)\, a|E\rangle$. Let $|E_0\rangle$ be the state with smallest energy, called the vacuum state. Then by definition, $a|E_0\rangle = 0$. If we apply the operator $\hbar\omega a^\dagger$ to this expression, we get:

$$\left(H - \hbar\omega\frac{1}{2}\right)|E_0\rangle = 0\,,$$

by applying (A.5) in the last line. From this we conclude that $H|E_0\rangle = \frac{1}{2}\hbar\omega|E_0\rangle$. This means that the lowest possible energy for the harmonic oscillator is $E_0 = \hbar\omega/2$: this is the vacuum energy.

Now we prove by induction that $H(a^\dagger)^n|E_0\rangle = E_n(a^\dagger)^n|E_0\rangle$, where $E_n = \hbar\omega(n+\frac{1}{2})$, for $n = 0, 1, 2\dots$. The base step for $n = 0$ was just proven. Now suppose that $H(a^\dagger)^{n-1}|E_0\rangle = E_{n-1}(a^\dagger)^{n-1}|E_0\rangle$. We will prove the equality for n. Remember that $Ha^\dagger = [H, a^\dagger] + a^\dagger H$ and $[H, a^\dagger] = \hbar\omega a^\dagger$, meaning that $Ha^\dagger = \hbar\omega a^\dagger + a^\dagger H$.

$$
\begin{aligned}
H(a^\dagger)^n|E_0\rangle &= Ha^\dagger\left((a^\dagger)^{n-1}|E_0\rangle\right) \\
&= \hbar\omega a^\dagger(a^\dagger)^{n-1}|E_0\rangle + a^\dagger H(a^\dagger)^{n-1}|E_0\rangle \\
&= \hbar\omega(a^\dagger)^n|E_0\rangle + \hbar\omega\left(n-\frac{1}{2}\right)(a^\dagger)^n|E_0\rangle = \hbar\omega\left(n+\frac{1}{2}\right)(a^\dagger)^n|E_0\rangle
\end{aligned}
$$

So the claim is true. Moreover, if the eigenstate $|E_n\rangle$ is normalised to unity, then $|E_{n+1}\rangle = a^\dagger|E_n\rangle/(n+1)^{1/2}$ is normalised as well:

$$
\langle E_{n+1}|E_{n+1}\rangle = \frac{1}{n+1}\langle E_n|aa^\dagger|E_n\rangle
$$

Now use the relation $\mathbb{1} = aa^\dagger - a^\dagger a$, so that $aa^\dagger = \mathbb{1} + a^\dagger a = \mathbb{1} + N$, hence:

$$
\langle E_{n+1}|E_{n+1}\rangle = \frac{1}{n+1}\left(1 + \left\langle E_n\middle|\frac{1}{\hbar\omega}H - \frac{1}{2}\middle|E_n\right\rangle\right) = 1 \tag{A.6}
$$

So the claim is proven. Similarly, one can show that $|E_{n-1}\rangle = n^{-1/2}a|E_n\rangle$ is normalised as well.

To conclude, if we normalise the vacuum state $|E_0\rangle$, we can construct all normalised eigenstates corresponding to higher energy values by simply using the raiser operator on $|E_0\rangle$. The eigenvalues, the energy values of a single quantum harmonic oscillator, are then given by (II.4): $E_n = \hbar\omega\left(n+\frac{1}{2}\right)$. Important to point out is the fact that the ground state E_0 has a *non-zero* energy value. When developing the partition function for a system of an infinite amount of quantum harmonic oscillators, this will prove to be vital for obtaining the Dedekind eta function. Later on, this will have consequences for the asymptotic behaviour of the Fourier coefficients and therefore of the system itself. See Sections IV and V.

B Analytic continuation of the Riemann zeta function

The remarkable result that the sum over all positive integers yields $-\frac{1}{12}$ arises as a consequence of the use of certain techniques in complex analysis to extend the domain of a function to values for which it is not defined. More precisely, we say that the above result emerges from the technique of analytic continuation of the Riemann zeta function to parts of the domain where it is a priori not convergent. The Riemann zeta function ζ_R is a function of a complex variable s that takes on the values

$$
\zeta_R(s) = \sum_{n=1}^{+\infty}\frac{1}{n^s}\,. \tag{B.1}
$$

If the real part of s is greater than 1, then the series in (B.1) converges and the function ζ_R gets assigned the classical values of these series. For other values of s, it is not immediately clear if meaningful values can be assigned to ζ_R, as the series could diverge classically. A way of assigning these meaningful values of ζ_R to this part of the domain is by way of analytic continuation of ζ_R. This means that we extend the domain of the function so as to keep ζ_R analytic on the entire new complex domain. A common way of doing this is by deriving certain functional relations that the function in question has to satisfy in order to be analytic on the extended domain. We will give a proof of why

$$\sum_{n=1}^{+\infty} n = -\frac{1}{12}$$

by showing, using the functional equation for ζ_R, what the value for $s = -1$ should be in order for ζ_R to be analytic on the extended domain, as this specific value of s yields a result on the sum over all integers. We will not derive this functional equation however, but rather recover it from the work of Hardy in [3], who gave an elaborate proof of it using techniques from complex analysis.

Theorem B.1. *If $s = -1$, then the Riemann zeta function takes on the value $\frac{-1}{12}$, i.e.*

$$\zeta_R(-1) = \sum_{n=1}^{+\infty} n = -\frac{1}{12} \tag{B.2}$$

Proof. Hardy showed in [3] that the Riemann zeta function satisfies the following functional equation:

$$\zeta_R(1-s) = 2(2\pi)^{-s} \cos\left(\frac{1}{2}s\pi\right)\Gamma(s)\zeta_R(s)\,. \tag{B.3}$$

We observe that $s = 2$ yields the value for $\zeta_R(-1)$. We know $\Gamma(2) = 1$ and

$$\zeta_R(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}\,.$$

The right hand side then becomes $-\frac{1}{12}$, which proves the result. □

It is a natural question to ask whether this result has an actual meaning in a physical context, as it seems a priori doubtful that a regularization of a seemingly divergent series has physical nuances. We remark however that, for example, the considered system of an infinite amount of harmonic oscillators itself exhibits a sort of limiting behaviour, and thus it is only reasonable that these results need to be invoked in order to assign context to the physics behind it. As mentioned before, the context in which we are working forms a stepping stone to quantum field theory, in which a continuum of quantum harmonic oscillators is investigated. So as to make any progress in this physical setting, it is necessary to adopt these mathematical results. A further motivation for the applicability of this result is given by its extensive use in string theory as well as in eigenvalue problems of partial differential equations, see for instance [4].

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